TOPOLOGY

M.Sc., MATHEMATICS First Year

Semester – I, Paper-IV

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M.Sc., MATHEMATICS - TOPOLOGY

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FOREWORD

Since its establishment in 1976, Acharya Nagarjuna University has been forging ahead in the path of progress and dynamism, offering a variety of courses and research contributions. I am extremely happy that by gaining 'A^+' grade from the NAAC in the year 2024, Acharya Nagarjuna University is offering educational opportunities at the UG, PG levels apart from research degrees to students from over 221 affiliated colleges spread over the two districts of Guntur and Prakasam.

The University has also started the Centre for Distance Education in 2003-04 with the aim of taking higher education to the doorstep of all the sectors of the society. The centre will be a great help to those who cannot join in colleges, those who cannot afford the exorbitant fees as regular students, and even to housewives desirous of pursuing higher studies. Acharya Nagarjuna University has started offering B.Sc., B.A., B.B.A., and B.Com courses at the Degree level and M.A., M.Com., M.Sc., M.B.A., and L.L.M., courses at the PG level from the academic year 2003-2004 onwards.

To facilitate easier understanding by students studying through the distance mode, these self-instruction materials have been prepared by eminent and experienced teachers. The lessons have been drafted with great care and expertise in the stipulated time by these teachers. Constructive ideas and scholarly suggestions are welcome from students and teachers involved respectively. Such ideas will be incorporated for the greater efficacy of this distance mode of education. For clarification of doubts and feedback, weekly classes and contact classes will be arranged at the UG and PG levels respectively.

It is my aim that students getting higher education through the Centre for Distance Education should improve their qualification, have better employment opportunities and in turn be part of country's progress. It is my fond desire that in the years to come, the Centre for Distance Education will go from strength to strength in the form of new courses and by catering to larger number of people. My congratulations to all the Directors, Academic Coordinators, Editors and Lesson-writers of the Centre who have helped in these endeavors.

Prof. K.GangadharaRao
M.Tech.,Ph.D.,
Vice-Chancellor I/c
Acharya Nagarjuna University

M.Sc. – Mathematics Syllabus

SEMESTER-I

104MA24: TOPOLOGY

Unit-I: Metric Spaces: Definition and some examples, Open sets, Closed sets, Convergence, completeness and Baire's theorem, Continuous mappings. (Sections 9 to 13 of chapter 2)

Unit-II: Topological spaces: The Definition and some examples, Elementary Concepts, Open bases and open subbases, Weak topologies. (Sections 16 to 19 of chapter 3)

Unit-III: Compactness: Compact spaces, Products of spaces, Tychonoff's theorem and locally compact spaces, Compactness for metric spaces, Ascoli's theorem. (Sections 21 to 25 of chapter 4).

Unit-IV: Separation: T1-spaces and Hausdorff spaces, completely regular spaces and normal spaces, Urysohn's Lemma and the Tietze extension theorem. (Sections 26 to 28 of chapter 5).

Unit-V: The Urysohnimbedding theorem, Connected spaces, The components of a space (Section 29 of chapter 5 and sections 31 to 32 of chapter 6).

TEXT BOOK: Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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M.Sc DEGREE EXAMINATION

First Semester

Mathematics :: Paper IV-TOPOLOGY

MODEL QUESTION PAPER

Time: Three hours Maximum:70 Marks

Answer ONE question from each unit (5x14=70)

UNIT-I

1. (a) Prove that in any metric space ⍰, each open sphere is an openset.

   (b) State and prove Baire's theorem.

(or)

2. (a) State and prove Cantor's Intersection theorem.

   (b) Let ⍰ and ⍰ be metric spaces and f is mapping of ⍰ into ⍰. Then show that ⍰ is continuous at $⍰_{\emptyset}$ iff $⍰_n \to ⍰_{\emptyset} \Rightarrow ⍰(⍰_n) \to ⍰(⍰_{\emptyset})$.

UNIT-II

3. (a) State and prove Lindelof's theorem.

   (b) Let x be topological spaces. Then show that any closed subset of X is the disjoint union of its interior and its boundary. That it contains these sets, they are disjoint, and it is their union.

(or)

4. (a) Let f:x y be a mapping of one topological space into another, and let there be given as open set base in X and an open subbase with its generated open base in Y. Then show that (i) f is continuous if the inverse image of each sub basic open set open and (ii) f is open if the image of each basic openset is open.

   (b) Let X be a second countable space, Then show that any open base for X has a countable subclass which is also an openbase.

UNIT-III

5. State and prove Heine-Borel theorem.

(or)

6. State and prove Ascoli's theorem.

UNIT-IV

7. State and prove Urysohn's theorem.

(or)

8. (a) Prove that every compact subspace of Hausdorff space is closed.

   (b) Show that the product of any non-empty class of Hausdroff space is a Hausdroff space.

UNIT-V

9. State and prove Urysohn's imbedding theorem.

(or)

10. (a) Let $\square$ be a topological space. If $\{\square_i\}$ is non-empty class of connected subspace of $\square$ such that is $\cap_i \square_i$ non-empty, Then show that $\square = \cup_i \square_i$ is also connected subspace of $\square$.

    (b) Show that any continuous image of connected space is connected.

CONTENTS

LESSON-1

TOPOLOGY DEFINITION AND SOME EXAMPLES IN METRIC SPACES

OBJECTIVES:

- To introduce the notion of metric space.
- To illustrate the concept of metric space by means of examples.
- To obtain some related properties of metric spaces through examples.
- To introduce the concept of metric space and illustrate it by means of examples. This is the context of lesson 1.
- (i) To introduce the concepts of open sphere, and open set in metric space.
- (ii) To obtain the properties of open spheres, open sets in metric spaces and using these to characterize the open intervals in the space R of real numbers. These concepts are dealt in Lesson 2
- To introduce the concepts of closed sphere and closed sets in metric spaces and to derive their basic properties is the content of Lesson 3.
- To introduce the concept of convergence, completeness, Cauchy sequences in metric spaces and prove Baire theorem regarding sequences of closed sets in metric spaces. These are established in Lesson 4
- To introduce the concept of continuity and uniform continuity in metric spaces and to characterize continuity in terms of convergent sequences and open sets in metric spaces, which is the content of Lesson 5.

STRUCTURE:

1.1 INTRODUCTION:

In this lesson the concept of a metric space, which is a generalization of the space R of a real number, with the distance function defined by means a modulus of a real number, is introduced and examples of metric spaces from various known spaces are given. Further some interesting examples about metric spaces are given

The concept of distance is introduced into the set of real numbers through the notion of modules$|x|$ of a real of x, which is defined as $|x| \geq 0$ and $|x|$=0 if, and only if ,$x = 0$. This modulus function on the set R of real numbers satisfies

(i) $|x-y| \geq 0$ and $|x-y| - 0 \Leftrightarrow x - y$

(ii) $|x-y| = |y-x|$ and

(iii) $|x-y| \leq |x-z| + |z-y|$, for all $x, y, z \in R$

The concept of the notions of limit, convergence, continuity, differentiation and integration are introduced through this distance function.

The generation of modulus function and the distance function on the set of real numbers to are arbitatry set leads to the study of metric spaces. A set X with a metric d satisfying the properties akin to the distance function in the set of real numbers constitute metric spaces. In these metric spaces the concepts of limit, convergence, continuity are studied.

1.2: Definition and some examples:

Definition 1.2.1:

Let X be a non-empty set. A metric on X is a real function d of ordered pairs of elements of X which satisfies the following three conditions:

(i) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$ (symmetry)

(iii) $d(x,y) \leq d(x,z) + d(y,z)$ (the triangle inequality).

The function d assigns to each pair (x,y) of elements of X a non- negative real number d(x,y) which by symmetry does not depend on the order of elements; d(x,y) is called the distance between x and y. A metric space consists of two objects: a non-empty set X and a metric d on X. The elements of X are called the points of the metric space (X,d). Usually we will be uniformal and denote the metric space (X,d) by X itself. However one should keep in mind, that metric space means non-empty set together with a metric. One can different metrics on the same set, which make it into distinct metric spaces. The following example is rather trivial but shows that every non-empty set can be regarded and a metric space.

1.2.2: Examples:

Let X be an arbitrary non-empty set and defined by

$$d(x,y) = \begin{cases} 0 \; if \; x = y. \\ 1 \; if \; x \neq y. \end{cases}$$

Then prove that d is metric on X.

Solution: Let X be an arbitrary non-empty set and d a function defined on X, such that d: X$X \times X \rightarrow R$ defined by

$$d(x,y) = \begin{cases} 0 \; if \; x = y. \\ 1 \; if \; x \neq y. \end{cases}$$

(i) We have $d(x,y) \geq 0$, for all $x, y \in X$, Whether $x = y, or, x \neq y$.

Also $d(x,y) = 0 \Leftrightarrow x = y, since$ $d(x,y) = 1, for$ $x \neq y$

This proves the first metric property.

(ii) Again $d(x,y) = d(y,x) = 0 \; when \; x = y$
and $d(x,y) = d(y,x)$=1, for $x \neq y$.
In either case d(x,y)=d(y,x), for all $x, y \in x$
This proves the second metric property.

(iii) Further, if x=y, so that d(y,x) =0, we have

$d(x,y) \leq d(x,z) + d(z,y)$.

Since d(x,z) = 0 if x=z

= 1 if x≠z

Similarly d(z,y) = 0 if z = y

= 1 if z ≠ y

But in either case d(x,z)+d(z,y) is not less than zero.

Similarly when x ≠ y,

$$d(x,y) \leq d(x,z) + d(z,y).$$

This satisfies the fourth metric property.

Hence d defined on X is a metric.

1.2.3: Examples :

Consider the real line R and the real function $|x|$ defined R by:

$$|x| = \begin{cases} x \; if \; x > 0 \\ -x \; if \; x < 0 \\ 0 \; if \; x = 0 \end{cases}$$

Then show that a metric d defined on R as d(x,y) = $|x - y|$, for all $x, y \in R$ is actually metric.

Solution: Let R be the set of points on a real line and the real function $|x|$ be defined on R by

$$|x| = \begin{cases} x \; if \; x > 0 \\ -x \; if \; x < 0 \\ 0 \; if \; x = 0 \end{cases}$$

That is,$|x|$ denotes the modulus of x, which is always positive. We know that the modulus function satisfies :

(i) $|x| \geq 0 \; and \; |x| = 0 \Leftrightarrow x = 0$

(ii) $|-x| = |x|$;

(iii) $|x+y| \leq |x| + |y|$

Consider the d-function on R defined by $d(x,y) = |x-y|, for\ all\ x, y \in R.$

(a) Since $|x-y| \geq 0$, it follows that

$d(x,y) = |x-y| \geq 0.$

That is, $d(x,y) \geq 0.$

Also since $|x| = 0 \Leftrightarrow x = 0$, it follows that

$d(x,y) = |x-y| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$

This proves the first metric property.

(b) Since $|-x| = |x|$ it follows that $d(x,y) = |x-y| = |-(x-y)|$

$= |y-x| = d(x,y)$

That is d(x,y) = d(y,x)

This proves the second metric property.

(c) Further, since $|x+y| \leq |x| + |y|$, it follows that

$$
\begin{aligned}
d(x,y) &= |x-y| \\
&= |x-z+z-y| \\
&\leq |x-z| + |z-y| \\
&= d(x,z) + d(z,y)
\end{aligned}
$$

That is $d(x,y) \leq d(x,z) + d(z,y)$

This proves the third metric property.

Thus d is a metric on R. This is called the usual metric on R, or, the metric induced by the modulus.

1.2.4: Examples :

Show that the set $\mathbb{C}$ of complex numbers with d-function defined by

$d(z_1 - z_2) = |z_1 - z_2|, for\ all\ z_1, z_2 \in \mathbb{C}$ is a metric space.

Solution : Let $\mathbb{C}$ be the set of complex numbers with d- function defined by

$d(z_1 - z_2) = |z_1 - z_2|, for\ all\ z_1, z_2 \in \mathbb{C}$

Where $|z|$ denotes the modules of z, which is always positive and is given by

$|z| = \sqrt{x^2 + y^2}$, where z = x+iy.

The given function $d(z_1, z_2) = |z_1 - z_2|$ follows from the following properties of the modulus function $|z|$.

(i) $|z| \geq 0\ and\ |z| = 0 \Leftrightarrow z = 0$;

(ii) $|-z| = |z|$; and

(iii) $|z_1 + z_2| \leq |z_1| + |z_2|$

We now show that(c,d) is a metric space.

(a) From (i) it follows that

$$d(z_1, z_2) = |z_1 - z_2| \geq 0$$

And $d(z_1, z_2) = 0 \Leftrightarrow |z_1 - z_2| = 0 \Leftrightarrow z_1 = z_2$

This proves the first metric property

(b) From (ii) it follows that

$d(z_1, z_2) = |z_1 - z_2| = |-(z_1 - z_2)|$

$= |z_2 - z_1| = d(z_2, z_1)$

That is $d(z_1, z_2) = d(z_2, z_1)$

This proves the second metric property.

(c) From (iii), we have

$d(z_1, z_2) = |z_1 - z_2| = |z_1 - z_3 + z_3 - z_2|$

$\leq |z_1 - z_3| + |z_3 - z_2|$

$= d(z_1, z_3) + d(z_3, z_2)$

That is $d(z_1, z_2) \leq d(z_1, z_3) + d(z_3, z_2)$

This proves that third metric property.

Thus d is a metric on $\mathbb{C}$ and hence $(\mathbb{C}, d)$ is a metric space.

1.2.5: Example:

Let f be a real function defined on the closed unit interval [0,1]. We say that f is bounded function if there is a real number k such that.

$$|f(x)| \leq k, \text{ for every } x \in [0,1].$$

Let X denote the set of all bounded continuous function defined on the closed unit interval [0,1] on X we define addition '+' by

$(f + g)(x) = f(x) + g(x)$.

The zero function '0' is defined by

$0(x)=0, \forall x \in [0,1]$,

Is evidently bounded and continuous. So $0 \in X$.

The negative –f of $f \in X$ is defined by

$(-f)(x) = -f(x), x \in [0,1]$

One can easily see that $-f \in X$.

Since each $f \in X$, is bounded and continuous on [0,1], it is Riemann integrable over[0,1].

We define the norm $\|f\|$ of a function $f \in X$ by

$\|f\| = \int_0^1 |f(x)|\, dx$.

Evidently $\|f\| \geq 0, \forall f \in X$.

We define the d- function on X by :

$d(f,g) = \|f - g\| = \int_0^1 |f(x) - g(x)|\, dx$

we see that d is metric on X.

(a) $d(f,g) = \|f - g\| = \int_0^1 |f(x) - g(x)|\, dx$

≥ 0, since $|f(x) - g(x)| \geq 0, \forall x \in [0,1]$

Also $d(f,g) = 0 \Leftrightarrow \int_0^1 |f(x) - g(x)|\, dx = 0$

$$\Leftrightarrow |f(x) - g(x)| = 0$$

$\Leftrightarrow \|f - g\| = 0$

$\Leftrightarrow f = g.$

(b) d(f,g) = $\|f - g\| = \int_0^1 |f(x) - g(x)|\, dx$

$$= \int_0^1 |f(x) - g(x)|\, dx$$

$$= \|g - f\|$$

$$= d(g,f)$$

(c)For f,g,h $\in X$.

$$d(f,g) = \|f - g\| = \int_0^1 |f(x) - g(x)|\, dx$$

$$= \int_0^1 |f(x) - h(x) + h(x) - g(x)|\, dx$$

$$= \int_0^1 |f(x) - h(x)|\, dx + \int_0^1 |h(x) - g(x)| dx$$

$$= \|f - h\| + \|h - g\|$$

= d(f,h)+d(h,g).

So 'd' is metric on X and hence (X,d) is a metric space.

1.2.6: Example:

Let d be metric on a non-empty set X. Show that the function d_1 defined by

$$d_1(a,b) = \frac{d(a,b)}{1+d(a,b)},$$

For a,b $\in X$, is also a metric on X.

Solution : Let d be a metric on a non empty set X.

Let d_1 be a function defined by

$$d_1(a,b) = \frac{d(a,b)}{1+d(a,b)} \text{ ,for a,b} \in X$$

We now prove that d_1 is also a metric on X

(a) For a,b $\in X$,

$$d_1(a,b) = \frac{d(a,b)}{1+d(a,b)}, \geq 0, since\ d(a,b) \geq 0.$$

Also, $d_1(a,b) = 0 \Leftrightarrow \frac{d(a,b)}{1+d(a,b)} = 0$

$\Leftrightarrow d(a,b) = 0$

$\Leftrightarrow a = b.$

This shows that d_1 satisfies the first axiom of metric

(b) For a,b $\in X$

$d_1(a,b) = \frac{d(a,b)}{1+d(a,b)} = \frac{d(b,a)}{1+d(b,a)}$ (since d is metric on X)

$d_1(b,a)$

That is $d_1(a,b) = d_1(b,a)$.

This shows that d_1 satisfies the second axiom of a metric space.

(c) Let a,b,c $\in X$. Then

$$\frac{d(a,b)}{1+d(a,b)+d(b,c)} \le \frac{d(a,b)}{1+d(a,b)} = d_1(a,b) - - - - - 1$$

And

$$\frac{d(b,c)}{1+d(a,b)+d(b,c)} \le \frac{d(b,c)}{1+d(b,c)} = d_1(b,c) - - - - - 2$$

Since d is metric on X , we have
$d(a,c) \le d(a,b) + d(b,c).$

So $d_1(a,c) = \frac{d(a,c)}{1+d(a,c)} \le \frac{d(a,b)+d(b,c)}{1+d(a,b)+d(b,c)}$

(since d-value is positive).

$$= \frac{d(a,b)}{1+d(a,b)+d(b,c)} + \frac{d(b,c)}{1+d(a,b)+d(b,c)}$$

$$\le d_1(a,c) + d_1(b,c)$$

That is $d_1(a,c) \le d_1(a,b) + d_1(b,c)$

This proves that d_1 satisfies the third axiom of metric.

Thus d_1 is also a metric on X.

1.2.7: Remark:

After defining a metric d_1 on X with respect to the metric d on X by
$d_1(a,b) = \frac{d(a,b)}{1+d(a,b)},$
One ear inductively define a metric d_n on X by
$d_n(a,b) = \frac{d_{n-1}(a,b)}{1+d_{n-1}(a,b)},$

For every positive integer n. so, if X is a metric space, than it can be made into a metric space in an infinitely many ways.

1.2.8: Remark :

If d is a metric on X, Then the metric d_1 defined by $d_1(a,b) = \frac{d(a,b)}{1+d(a,b)}$, is such that $d_1(a,b) < d(a,b)$. So, the process of obtaining the metrics d_n,
From d is a distance decreasing process.

1.3 : Fundamental Principles relating to metric spaces:

1.3.1: Definition :

Let (X,d) be a metric space and let Y be are arbitrary non-empty subset of X. If the function d is considered to be defined only for the points of Y provided (Y,d) is evidently itself a metric space.

Then, Y, with d restricted in this way, is called a subspace of (X,d).

1.3.2: Example:

(i)The closed unit interval [0,1] is a subspace of the real line, as is the set consisting of all the rational points.
(ii)The unit circle, the closed unit disc, and the open unit disc are subspace of the complex plane.
(iii)The real line is itself is a subspace of the complex plane.

1.3.3: Definition:
An extended real number system is the real number system R together with the symbols $-\infty, and +\infty$ $such\ that -\infty < +\infty$ and for every real number X. $-\infty < x < +\infty$.

1.3.4 : Definition:

Let A be a non empty subset of R of real numbers. An element X in R is called a lower bound of A, if $x \leq a$ for each $a \in A$; and lower bound of A is called a greatest lower bound, or, infimum of A if it is greater than or equal to every lower bound of A and simply it is written as Inf A.

In other words, if A is non-empty and has a lower bound, then the greatest lower bound or infimum is the largest real number x such that $x \leq a$ for every a in A.

If A is non-empty and has no lower bound, we put inf A $=-\infty$ and if A is empty, we put Inf A $=+\infty$

1.3.5 : Definition:

Let A be non-empty set of R of real numbers. The least upper bound, or, supremum of the non-empty set A is smallest upper bound of A, simply it is written as Sup A.

If A is a non-empty set of real numbers which has no upper bound, and therefore no least upper bound in R, we express this by writing.

Sup A $=+\infty$

And if A is the empty subset of R, we put

Sup A = $-\infty$

1.3.6 : Remark :

(i) The first advantage of the extended real number system is that it enables sup A and inf A for subsets A of the real line with out any restrictions whatever on the nature of A.

(ii) The second advantage is the availability of the symbols $-\infty$ and $+\infty$ which lead to a reasonable extension of the concept of an interval on the real line.

1.3.7 : Definition :

Let a and b be any two real numbers such that a≤b. Then the closed interval from a to b is the subset of the real line R defined by

[a , b] = $\{x: a \le x \le b\}$

If b is real number and a is an extended real number such that a<b, then the open closed interval from a to b is

(a , b] = $\{x: a < x \le b\}$

So$(-\infty, b]$ is also an open closed interval.

If a is real number and b is an extended real number such that a<b, then the closed open interval from a to b is

[a , b) = $\{x: a \le x < b\}$

So [a , $+\infty$) is also a closed open interval.

If a and b are extended real numbers such that a<b, Then the open interval from a to b is

(a , b) = $\{x: a < x < b\}$

This adds to the previously defined open intervals those of the form$(-\infty, b)$ where b is real,$(a, +\infty)$ where a is real and $(-\infty, +\infty)$ thought our study, the term interval will always signify one of the four types (a , b),

1.3.8 :Definition :

Let X be a metric space with metric d, and let A be a subset of X. If x is a point of X, then the distance from x to A is defined by ;

$d(x, A) = inf\{d(x, a): a \in A\}$

that is, the distance from x to a is the greatest lower bound of the distances from x to points of A.

1.3.9 :Definition :

Let X be a metric space with metric d, and let A be a subset of X. The diameter of the set A is defined by

$$d(A) = sup\{d(a_1, a_1): a_1 \ and \ a_2 \in A\}$$

The diameter of A is thus the least upper bound of the distances between pair of its points. A is said to have finite diameter or infinite diameter according as d(A) is a real number, or ,$\pm\infty$.

The empty set has infinite diameter, since d$(\emptyset) = -\infty$

A bounded set is one whose diameter is finite.

1.3.10 :Definition :

A mapping of a non empty set into a metric space is called a bounded mapping if its range is bounded set.

Excercise :

(i)Let X be a non-empty set and let d be a real function of ordered pairs of elements of X, which satisfies the conditions:

(i)$d(x,y)=0 \Leftrightarrow x=y$

(ii)$d(x,y) \leq d(x, z)+d(y, z)$.

show that d is a metric on X

(ii)Let X be the set of bounded continuous functions defined on [0,1]. Define another $\|.\|$on X by :

$\|f\| = sup\{|f(x)| \,/ x \in [0,1]\}$,

Which we briefly write is

$\|f\| = sup|f(x)|$.

Next define the d-function on X by

$$d(f,g) = \|f - g\|$$

Show that d is a metric on X.

(iii)Let $x_1, x_2, x_3, x_4, \ldots\ldots\ldots, x_n$ be a finite class of metric spaces with metric $d_1, d_2, d_3, \ldots\ldots d_n$ respectively and let

$$X = X_1 \times X_2 \times X_3 \times \ldots\ldots \times X_n = \{(x_1, x_2, \ldots\ldots x_n) / x_i \in X_i, 1 \leq i \leq n\}$$

Show that each of the following functions d and $\overline{d}$ are metrics on X :

(i)$d((X_1, X_2, X_3, \ldots\ldots X_n), (Y_1, Y_2, Y_3, \ldots\ldots Y_n)) = max\{d_i(x_i, y_i) / 1 \leq i \leq n\}$;

(ii)$\overline{d}((X_1, X_2, X_3, \ldots\ldots X_n)(Y_1, Y_2, Y_3 \ldots\ldots Y_n)) = \sum_{i=1}^{n} d_i(x_i, y_i)$

1.4 MODEL EXAMINATION QUESTIONS:

(i)Define a metric space and illustrate it by means of an example.

(ii)Let X be an arbitrary non-empty set, show that the function d defined by

$$d(x,y) = \begin{cases} 0 \text{ } if\ x = y, \\ 1 \text{ } if\ x \neq y. \end{cases}$$

For all $x, y \in R$, is a metric on X.

(iii)Show that the set R of real numbers with d function defined by

$d(x,y) = |x - y|$, for all $x, y \in R$ is a metric space.

(iv)Show that the set of $\mathbb{C}$ complex numbers with d function defined by

$d(z_1, z_2) = |z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$

(v)Let X be a metric space with metric d. show that d_1, defined by

$d_1(x, y) = \frac{d(x,y)}{[1+d(x,y)]}$ is also a metric on X.

1.5 SUMMARY:

A metric space is a set where a distance function is defined, allowing you to measure the distance between any two elements within that set, with the key property that this distance function must satisfy specific axioms like non-negativity, symmetry, and the triangle inequality essentially. It formalizes the concept of distance in a general setting.

1.6 TECHNICAL TERMS:

- Metric Space: A set X together with a function d: X $\times$ X$\rightarrow$ R that satisfies the properties of a metric.
- Discrete Metric Space: A metric space where the distance between any two distinct points
- Compact Metric Space: A metric space that is compact, meaning that every sequence has a convergent subsequence.
- Complete Metric Space: A metric space where every Cauchy Sequence Convergence.

1.7 SELF ASSESSMENT QUESTIONS:

1.What is meant by metric space?
Ans: A metric space is defined as a non-empty set with a distance function connecting two metric points.

2.Given an example of metric space?
Ans: The well-known example of metric space is the set R of all real numbers with p(x, y) = | x – y |.

3.What is the triangle inequality property for the metric?
Ans: The triangle inequality property for the metric is given by: p(x, y) ≤ p(x, z) + p(z, y).

4.Explain the difference between the metric and the norm.
Ans: A metric measures the distance between two places in space, whereas a norm measures the length of a single vector. A metric can be defined on any set, while a norm can only be specified on a vector space.

1.9 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Dr. M. Gnaneswara Reddy

LESSON-2

OPEN SETS AND CLOSED SETS IN METRIC SPACES

OBJECTIVES :

- To introduce the concepts of open spheres, interior point of set and open sets in metric spaces.
- To illustrate the above concepts by means of suitable examples.
- To show that every open sphere ia an open set in a metric space.
- To obtain basic properties of open sets in metric space.
- To establish that every open set in a metric space ia a union of open spheres.
- To prove that every open set in the space R of real numbers is a countable union of disjoint open spheres.
- To introduce the notion of interior of a subset of a metric space and relate it to the notion of open sets.
- To introduce the concept of limit point of a set in a metric space.
- The notions of limit point, closed sphere and closed sets are illustrated by means of examples.
- To prove that every closed sphere is a closed set in a metric space.
- To obtain the basic properties of closed sets.

STRUCTURE:

2.1 INTRODUCTION :

The notions of the open sphere, open set limit point of a set, closed sphere and closed set are introduced in a metric space and its basic properties are obtained in this lesson. The open sets in metric space are charecterised by means of open spheres. The complementary property of open sets and closed sets is established. Further the open sets in the space R of real numbers are also charecterised through open intervals.

2.2 OPEN SETS AND METRIC SPACES:

2.2.1 : Definition :

Let X be a metric space with metric d. Let x_0 be a point of X and let r be a positive real number. Then open sphere $S_r(x_0)$ with center x_0 and radius r is the subset of X defined by

$$S_r(x_0) = \{x \in X \,/\, d(x, x_0) < r\}$$

2.2.1 : Remark :

Since $d(x,x_0) = 0 < r, x_0 = S_r(x_0)$ and thus an open sphere is always non-empty set, and it contains its center x_0.

2.2.2 : Example :

In the metric space R of real numbers with respect to the usual metric, the open sphere with center X_0 radius r is given by

$$s_r(x_0) = \{x \in R / \ d(x,x_0) < r\}$$

$$= \{x \in R / |x, x_0| < r\}$$

$$= \{x \in R / x_0 - r < x < x_0 + r\},$$

which is the open interval (x_0-r, x_0+r).

2.2.3 Example :

For any x_0 in the metric space (x, d), where the metric d is defined by

$$d(x,y) - \begin{cases} 0 \ if \ x = y \\ 1 \ if \ x \neq y, \end{cases}$$

Then $S_1(x_0)=1=\{x_0\}$ and $S_r(x_0)=X$, for every $r > 1$.

Solution :

The open sphere $S_1(x_0) = \{x \in X \ / \ d(x,x_0) < 1\} = \{x_0\}$ so the open sphere is the singleton set $\{x_0\}$and thus it contains only x_0.

Suppose r >1. Consider $s_r(x_0)$ clearly $s_r(x_0) \subseteq X - - - - - 1$

Let any$x \in X$ if $x = x_0$ then $d(x,x_0) = d(x_0,x) = 0 < 1 < r$. If $\neq x_0$, then $d(x,x_0) = 1 < r$, so that. That is $x \in s_r(x_0)$. That is $X \subseteq s_r(x_0) - - - -(2)$

From (1) and(2) $s_r(x_0) = X$.

2.2.4 : Definition :

Let G be a subset of a metric space. A point x of G is called interior point of G if there exist an open sphere $s_r(x)$such that $s_r \ (x) \subseteq G$

The set of all interior points point of G is called the interior of G and it is denoted by Int(G).

2.2.5 : Definition :

A subset G of a metric space is said to be an open set if every point of G is an interior point of G. That is, G is an open set , if for every x in G, there exist an open sphere $s_r(x)$ such that $s_r(x) \subseteq G$.

2.2.6 : Example :

The open interval (a ,b) of the real line R is an open set.

Solution :

Let $x \in (a,b)$ choose r such that 0<r< min{x-a, x-b}.

Then $s_r(x) = (x-r, x+r) \subseteq (a,b)$.

So every point of (a,b) is an interior point of (a,b) and thus open interval (a,b) is an open set in R.

2.2.7 : Example : The subset [0,1) is not an open set in R.

Solution : For any real number r>0 (however small r may be), the open sphere $s_r(0) = (-r, r)$ contains infinitely many points of (-r, 0), which do not belong to [0, 1], so that $s_r(0) \nsubseteq [0,1)$. So $0 \in [0,1)$ is not an interior point of [0, 1) and thus [0,1) is not an openset in R.

2.2.8 : Theorem :

In any metric space X, the empty set $\emptyset$, and the full space X are open sets.

Proof : Since $\emptyset$ has no points, the fact that every point of $\emptyset$ is an interior point of $\emptyset$ is trivially satisfied. Coming to the full space X, if $x \in X$ and $s_r(x)$ is are open sphere in X, center x and radius r>0, then $s_r(x)$ is trivially a subset of full space X, so that $s_r(x) \subseteq X$. Thus x is an interior point of X and X is open.

2.2.9 : Remark :

In Example 2.3.7, we have seen that [0, 1) is not open as a subset of the real line R. But if we consider [0, 1) as a metric space X in its own right, [as a subspace of the real line], then [0, 1) is open itself. Thus a set is open, or, not open only with respect to a specific metric space containing it, never on its own.

2.2.10 : Theorem :

In any metric space X, each open sphere is an open set.

Proof : Let (x,d) be metric space and let $x_0 \in X$, consider the open sphere $s_r(x_0)$. To show that $s_r(x_0)$ is open in (X,d), we have to show that every point of $s_r(x_0)$ is an interior point of $s_r(x_0)$.

To see this, let $x \in s_r(x_0)$.

Choose $\in > 0$ such that $\in = r - d(x, x_0)$. Then for every $y \in s_\in(x)$,

We have $d(x,y) < \in$ and thus,

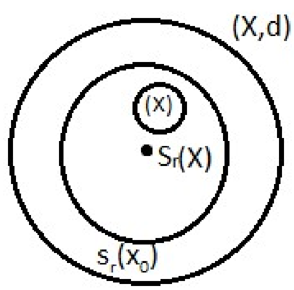
$d(x_0,y) \leq d(x_0,x)+d(x,y)$

$< d(x_0,x)+\in$

$= d(x_0,x)+r-d(x,x_0)$

$= r$

That is $d(x_0,y) < r$ and $y \in s_r(x_0)$.



That is, $s_\in(x) \subseteq s_r(x_0)$ and thus x is an interior point of $s_r(x_0)$ So $s_r(x_0)$ is an open set. The following theorem characterizes open sets in terms of open spheres.

2.2.11 : Theorem :

Let (X, d) be a metric space. A subset G of x is open if, and only if, it is a union of open spheres.

Proof : Suppose that G is open. then every point of G is an interior point of G. So, for every $x \in G$,there exists an open sphere $S_{r_x}(x)$ such that $S_{r_x}(x) \subseteq G$.

Let $\{S_{r_x}(x)\}_{x,g}$ be the collection of all such open sphere.

Then, $\cup_{x\in G} s_{r_x}(x) \subseteq G$.

Further $x \subset s_{r_x}(x)$, so that G = $\cup_{x\in G}\{x\} \subseteq \cup_{x\in G} \subset s_{r_x}(x)$. That is G = $\cup_{x\in G} s_{r_x}(x)$.

Conversely, assume that G is a union of a class S of open spheres. We show that G is open.

If S is empty. Then G is also empty and hence it is open.

Suppose S is non-empty. then G is also non-empty. Let x $\in$ G then G being a union of sets in S,$x \in s_r(x_0)$, for some open sphere $s_r(x_0) \in S$. Since every open sphere is an open set. $x \in s_r(x_0)$ implies x is an interior point of $s_r(x_0)$. so, there exists an open sphere $s_{r_1}(x)$ such that $s_{r_1}(x) \subseteq s_r(x_0)$.Then

$s_{r_1}(x) \subseteq s_r(x_0) \subseteq U\{s_\alpha / s_\alpha \in S\} = G$.

So, x is also an interior point of G and thus G is an open set.

The next theorem gives the fundamental properties of open sets in metric spaces.

2.2.12 : Theorem :

Let X be a metric space. Then

(i) any union of open sets in X is open ; and

(ii) any finite intersection of open sets in X is open.

Proof : Let X be metric space.

(i)Let $\{G_\alpha\}$ be any class of open sets in X and G = $\cup_\alpha G_\alpha$.

We shall show that G is open in X.

To see this let $x \in G$.

Then $x \in G_\alpha$, for some α. Since G_α is open in X, there exists an open sphere $S_r(x)$ such that $S_r(x) \subseteq G_\alpha$. But $G_\alpha \subseteq \cup_\alpha G_\alpha = G$. so $s_r(x) \subseteq G$.

That is, x is an interior point of G and thus G is open.

(ii)Let $\{G_\alpha\}_{i=1}^n$be any finite class of open sets in X and let G = $\cap_{i=1}^n G_i$.

To show that G is open in X, Let $x \in G$

Then $x \in G_i$, for every i.

Since for each i , $1\leq i \leq n$, G_i is open, there exists open spheres $s_{r_i}(x), 1 \leq i \leq n$, such that $s_{r_i}(x) \in G_i$.

Let r = min $\{r_1, r_2, \ldots\ldots r_n\}$.Then $s_r(x) \subseteq s_{r_i}(x), for\ 1 \leq i \leq n$ (since they have common center and $r \leq r_i$).

So $s_r(x) \subseteq G_i, for\ 1 \leq i \leq n, or, s_r(x) \subseteq \bigcap_{i=1}^{n} G_i = G$.

Thus x is an interior point of G and hence G is open.

2.2.13 : Remark :

The condition of "finite ness" in the above theorem is essential. To see this, consider the infinite class$\left(-\frac{1}{n},\frac{1}{n}\right)$ of open intervals on the real line.

Here each $\left(-\frac{1}{n},\frac{1}{n}\right)$ is open. But $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$=$\{0\}$ and $\{0\}$ is not an open subset in R, since every open sphere (-r,r), r>0 contains infinitely many points not in $\{0\}$.

The following theorem gives the charecterstic property of an open set in the set R of real numbers.

2.2.14 : Theorem :

Every non-empty open set of the set R of real numbers is the union of a countable disjoint class of open intervals.

Proof : step 1 :

Let G be an open set in R and Let $x \subset G$. Then there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subset G$. So, There is a y>x(that is ,any y such that $x<y<x+\delta$ has this property) such that $(x, y) \subset G$.Let b = sup$\{y/(x,y) \subset G\}$.

Similarly, there is a z<x such that $(z , x) \subset G$. (Any z, such that $x-\delta < z < x$ has this property).Let a = inf$\{z/(z , x) \subset G\}$. Then a<x<b

So I_x = (a , b), is an open interval containing x.

Step -2 :

We shall $I_x \subset$ G. To see this, Let $\in I_x$, such that x<w<b. Since b =sup $\{y/(x,y) \subset G\}$, there is real number y such that w<y and $(x,y) \subset G. so\ w \in G$.

More over $b \notin G$.For if $b \in G$, then G is open set implies there is an $\in > 0$ such that $(b-\in, b+\in) \subset G$, when $(x, b+\in) \subset G$, This contradicts the supremum property of b. so $b \notin G$.

Similarly if w is such that a<w<x, by using the fact that a = inf $\{z/(z,x) \subset G\}$ one can show that $w \in G$ and $a \notin G$ Thus $I_x \subset G$.

Step -3 :

Consider the collection of open intervals $\{I_x / x \in G\}$. For every $x \in G$, I_x = (a , b) and a<x<b, where a = inf $\{z/(z, x) \subset G\}$

b = sup $\{x/(x, y) \subset G\}$

so, $x \in I_x$ and $G \subseteq \bigcup_{x \in G} I_x$.

Further, each $I_x \subset G$, so that $\bigcup_{x \in G} I_x \subseteq G$.

Hence G = $\bigcup_{x \in G} I_x$.

Step -4 : we now show that any two different intervals in $\{I_x / x \in G\}$ are either disjoint, or, identical.

Let (a, b) and (c, d) be any two different intervals in $\{I_x / x \in G\}$.
If they are disjoint then there is nothing to prove.
otherwise these intervals have a common point, say, e.
That is $a \le e \le b$ and $c \le e \le d$.
Then $e \le b$ *and* $c \le e$ imply that c<b.
Also a<d

In step 2, we have seen that c does not belong to G. Since G = $\bigcup_{x \in G} I_x$ and (a , b) = $\{I_x / x \in G\}$, it follows that $c \notin (a, b)$. so we must have $c \le a$.
Again a does not belong to G (by step 2), and (c , d) $\subseteq G$.so a does not belong to (c , d), so that $a \le c$.

Hence a=c.

Similarly, we can show that b=d, and thus (a,b) = (c,d).
Thus two different intervals in the collection $\{I_x / x \in G\}$ are either disjoint or identical.

Step -5 : We shall now show that the collection $\{I_x / x \in G\}$ is a countable collection.

Consider any I_x. I_x is an open interval. We know that any open interval contains a rational number r_x(by Archemedian property).

If r_x and r_y are rational numbers in the disjoint intervals I_x and I_y, then $r_x \ne r_y$. consider the set $\{r_x\}$ is rational number in $I_x = \{I_x / x \in G\}$.Then $\{r_x\}$is a subset of the set of rational numbers. Since the set of rational numbers is countable and hence $\{I_x / x \in G\}$ is countable.

These show that G = $\bigcup_{x \in G} I_x$,which is union of collection $\{I_x / x \in G\}$ of disjoint open interval I_x.

2.2.15 : Definition : (Interior of a set)

Let X be a metric space. The set of all interior points of A is called the interior of A and it is denoted by Int(A).

The basic properties of interiors are the following:

(i) Int (A) is an open set of A which contains every open subset of A.

(ii) A is open if, and only if, A=Int(A).

(iii) Int(A) equals the union of all open subsets of A.
In the following example we prove (ii).

2.2.16 : Example :

In a metric space X, a subset A of X is open if, and only if A=Int(A).

Solution : Let X be a metric space and let A be a subset of X.

First, Let us assume that A is an open set implies that A is a neighbourhood of each of its points implies that every point of A is an interior point of A .Implies that $A \subseteq Int\ A ----1$ Also each interior point of A belongs to A. Implies that $Int\ A \subseteq A ----2.$

From (1) and (2), we have A=Int(A).
Conversly, Let us assume that A=Int (A) implies that A is an open set, since Int (A) is an open set.

Hence A is an open set if, and only if, A=Int(A).

2.2.17 : Example :

Let X be a metric space, and show that any two distinct point of X can be saperated by open spheres.

Solution : Let X be metric space and let $x, y \in X\ and\ x \neq y.$

Put d(x, y)=d and choose a real number r such that $0 < r < \frac{d}{2}.$

Consider the open sphere $s_r(x)$ and $s_r(y)$.

For any $z \in s_r(x),$ we have $d(z,y) < r < \frac{d}{2}$

Suppose $z \in s_r(x),$ then d(x,z) < $\frac{d}{2}$

So $d(x,y) \leq d(x,z) + d(z,y) < \frac{d}{2} + \frac{d}{2} = d.$

This is a contradiction to the fact that d(x, y)=d. so $z \notin s_r(x).$

Similarly, one can show that for any $u \in s_r(x)$ and $u \notin s_r(x).$

Hence, $s_r(x) \cap s_r(y) = \emptyset$. That is, $s_r(x)$ and $s_r(y)$ are open spheres centered at x and y such that $s_r(x) \cap s_r(y) = \emptyset$

2.3: CLOSED SETS AND METRIC SPACE:

2.3.1 : Definition :

Let X be a metric space with metric 'd' and let $A \subseteq X$. A point x of X is called a limit point of A if each open sphere centered on x contains at least one point of a different from x.

2.3.2 : Example :

For the set A = $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \ldots\ldots\right\}$ in the real line R,0 is a limit point.

Solution :

To see this $\in > 0$.Then $s_{\in}(0) = (-\in, \in)$. If we choose a positive integer N such that $N > \frac{1}{\in}$ (or,$\frac{1}{N} < \in$). then $\frac{1}{N} \in A$ and also $\frac{1}{N} \in s_{\in}(0)$ (since $\left|0 - \frac{1}{N}\right| = \frac{1}{N} < \in$).so '0' is a limit point of A.

2.3.3 : Remark :

Observe that no other point of A is a limit point of A. That is, A has only one limit point 0 and this does not belong to A. This example also shows that a limit point of a set need not be in the set.

2.3.4 : Example :

The closed - open interval [0, 1) has 0 and 1 as limit points. Further every real number x such that 0<x<1 is also a limit point of this set. Observe that the limit point 1 does not belong to [0,1).

(proof is similar to that of example 2.3.2).

2.3.5 : Example :

The set of integers has no limit points.

Solution : Let n be any integers. For any real number 0<r<1,the open sphere (n-r, n+r) does not contain any integer except n. So n is not a limit point of the set $\{0, \pm 1, \pm 2, \ldots\ldots.. \pm n, \ldots..\}$ of integers. This shows that the set of integers has no limit points.

2.3.6 : Example :

For the set of rational numbers every real number is a limit point.

Solution :

Let Q and R respectively denote the set of rational numbers and the set of real numbers.

Let r be any real number and $\in > 0$ be any real number, however small. Then the open sphere $(r - \in, r + \in)$ contains infinitely many rational numbers since there are infinitely many rational numbers between two distinct real numbers. so, r is a limit point of the set Q of rational numbers.

2.3.7 : Definition :

A subset F of the metric space X is called a closed set, if it contains each of its limit points.

2.3.8 : Example :

The closed-open interval [0,1) is not a closed set in the set of real numbers, since it does not contain its limit point1, even though it contains all other limit points. (see example 2.4.4)

2.3.9 : Example :

The closed interval [0,1] is a closed subset of the set of real numbers, since it contains all its limit points including 0 and 1.

2.3.10 : Theorem :

In any metric spaces X, the empty set ϕ and the full space X are closed sets.

Proof: The empty set ϕ has no points and hence it has no limit points, so that the set of the limit points of the empty set ϕ is the empty set ϕ. Since $\emptyset \subseteq \emptyset$. trivially, ϕ is closed

The fully space X contains all the points of X. So it is automatically contains all its limit points and thus X is closed.

2.3.11 : Theorem :

Let X be a metric space. A subset F of X is closed if, and only if, its compliment F1 in X is open.

Proof : : Let F be a closed subset of a metric space X, and let F1 be its compliment in X. If F^1 is empty, then F^1 is trivially open.

So, let F^1 be non-empty and let $x \in F^1$. We shall show that x is an interior point of F^1, So that F^1 is open. Now $x \in F^1$ implies that $x \notin F$. Since F is closed and $x \notin F$, it follows x is not a limit point of F. So, there exists an open sphere, say, $s_r(x)$ such that $s_r(y)$ does not contain any point of F, or, $s_r(x) \cap F = \phi$. This means that $s_r(x) \subseteq F^1$ showing that x is an interior point of F^1 and thus F^1 is open.

On the otherhand, let F^1 be an open set in X. To show that F is closed in X, it is enough to show that F contains all its limit points. To see this ,let x be a limit point of F. If possible, let $x \notin F$. Then $x \in F^1$. Since F^1 is open, $x \in F^1$ implies that x is an interior point F^1. So, there exists an open sphere $S_r(x)$, such that $S_r(x) \subseteq F^1$. Then $S_r(x) \cap F = \emptyset$.

That is, there is an open sphere $s_r(x)$, centered x, which does not contain any point of F. This shows that x is not a limit of F. This is contradiction to the .So, F is closed. assumption that x is a limit point of F. So, our assumption that x does not belong to F is wrong and thus $x \in F$. So, F is closed.

Just as the notion of open sphere plays a key role in the charecterisation of open sets in metric spaces, the notion of closed sphere plays a similar role in the study of closed sets in metric spaces.

2.3.12 : Definition :

Let (X,d) be a metric space, r>0 a real number and $x_0 \in X$.The set $s_r[x_0] = \{x \in X / d(x, x_0) \leq r\}$ is called the closed sphere center x_0 and radius r in X.

Analogous to the theorem 2.2.10 for open spheres, we have the following theorem for closed spheres.

2.3.13 : Theorem :

In any metric space, each closed sphere is a closed set.

Proof : Let $s_r[x_0]$ be a closed sphere in X. To show that $s_r[x_0]$ is closed. It suffices to prove that its compliments $s_r[x_0]^1$ is open.

If $s_r[x_0]^1$ is empty, then trivially it is open. So, let $s_r[x_0]^1$ be non-empty and let $x \in s_r[x_0]^1$. Then $d(x,x_0)>r$. Let $r_1 = d(x,x_0)-r$. clearly $r_1>0$.Consider the open sphere $s_{r_1}(x)$.

We shall show that $s_{r_1}(x) \subseteq s_{r_1}[x_0]^1$, so that x is an interior point of $s_r[x_0]^1$ and thus it is open in X.

Let $y \in s_{r_1}(x)$, so that $d(x,y)<r_1$.

Now $d(x_0,x) \leq d(x_0,y)+d(x,y)$

So, $d(x_0,y) > d(x_0,x)-d(x,y)$

$>d(x_0,x) - r_1$, (Since $- d(x,y)-r_1$)

$= d(x_0,x)-[d(x,x_0)-r] = r$

So $y \notin s_r[x_0]$, so that $y \subset s_{r_1}[x_0]^1$. That is $s_{r_1}(x) \subseteq s_{r_1}[x_0]^1$ and x is an interior point $s_{r_1}[x_0]^1$. This shows that $s_{r_1}[x_0]^1$ open, so that $s_r[x_0]$ is closed.

The following theorem gives the main properties of closed sets.

2.3.14 : Theorem : Let X be metric space. Then
(i) any intersection of closed sets in X is closed.
(ii) Union of finite number of closed sets in X is closed.

Proof : Let X be a metric space.

(i) Let $\{C_\alpha\}$ be an arbitrary collection of closed subsets of X and let C = $\bigcap_\alpha C_\alpha$.
To show that C is closed set in X, it is sufficient to prove that C^1 is open in X.
From the De Morgan Laws, $(\bigcap_\alpha C_\alpha)^1 = \bigcup_\alpha C_\alpha^1$.

Since each C_α is closed in X, it follows that each C_α^1 is open in X. But arbitrary union of open sets is open. So that $\bigcup_\alpha C_\alpha^1$ is open. That is, $(\bigcap_\alpha C_\alpha)^1$ is open and thus $\bigcap_\alpha C_\alpha$ is closed. That is C = $\bigcap_\alpha C_\alpha$ is closed.

(ii) Let $\{C_i/1,2,3,\ldots\ldots,n\}$ be a finite collection of closed sets in X and let C= $\bigcup_{i=1}^n C_i$.

Again from the Demorgan laws, $[\bigcup_{i=1}^n C_i]^1 = \bigcap_{i=1}^n C_i^1$. Since each C_i is closed in X, it follows that each C_i^1 is open in X. But the intersection of finite number of open sets in X is open X, so that $\bigcap_{i=1}^n C_i^1$ is open in X. That is $[\bigcup_{i=1}^n C_i]^1$ is open in X, or, $\bigcup_{i=1}^n C_i$ is closed in X. Thus C= $\bigcup_{i=1}^n C_i$ is closed in X.

2.3.15 : Definition :

Let X be a metric space and let A be a subset of X. The closures of A is the union of A and the set of all its limit points and it is denoted by $\bar{A}$. The main facts about closures are the following.

(i) A is a closed super set of A, which is contained in every closed super set of A.
(ii) A is closed if, and only if, A=$\bar{A}$.
(iii) A equals the intersection of all closed supersets of A. In the following example we

prove (ii).

2.3.16 : Example : Let X be a metric space. A subset A of X is closed if, and only if A=$\bar{A}$.

Proof : Let X be a metric space.

Let D(A) be the set of limit points of A. Then by the definition of closure of A,

$$\bar{A} = A \cup D(A)$$

Suppose A is closed. Then A contains all its limit points, that is D(A)$\subseteq$ A. So $\bar{A} = A \cup D(A) = A$.

On the other hand let $\bar{A} = A$. Then A= $A \cup D(A)$, so that D(A)$\subseteq$ A, or, A contains all its limit points. So A is closed.

Hence A is subset A of X is closed if, and only if, A=$\bar{A}$.

Another concept that is related to closed sets in a metric space is the boundary of a set in a metric space.

2.3.17 : Definition :

Let X be a metric space and A a subset of X. A point in X is called a boundary point of A if each open sphere centered on the point intersects both A and A^1 .

The boundary of A is the set of all its boundary points one can prove easily the following properties of the boundary of a set A in a metric space X.

(i) The boundary of A equals to $\bar{A} \cap \overline{A^1}$

(ii) The boundary of A is closed set.

(iii) A is closed if, and only if, it contains its boundary.

2.3.18 : Example :

Let X be a metric space and let A be a sunset of X. If x is a limit point of A, show that each open sphere centered on x contains an infinite number of distinct points of A.

Solution : Let A be a subset of a metric space X and let x be a limit point of A. Suppose that there is an open sphere $S_r(x)$ which contains only a finite number of points of A, say $x_1, x_2, x_3, \ldots\ldots x_n$.

For $1 \leq i \leq n$, let r_i= d(X,X_i) and let r_0 be any real number such that 0 < r_0 < min {$r_1, r_2, r_3, \ldots\ldots r_n$}.

Clearly the open sphere or s_{r_0}(x) does not contain any point of A other that x. This contradicts the fact that x is a limit point of A. So, s_r(x) must contain an infinite number of distinct points of A.

2.3.19 :Example : Show that the finite subset of a metric space is closed.

Solution : Let A be a finite subset of a metric space X and let x $\in$ X.

If x is a limit point of A, then every neighborhood of x must contain an infinitely many distinct points of A. This is not possible, since A is finite. So x is not a limit point of A. This shows that the set of limit points of A. This shows that the set of limit points of A in X is the empty set ϕ.Since $\phi \subseteq$ A trivially, A is closed.

2.4 EXERCISES :

1. Let X be a metric space and A be a subset of X. Prove that the closure $\bar{A}$ of A is a closed superset of A which is contained in every closed superset of A.
2. Let X be a metric space and A be a subset of X. Prove that the closure $\bar{A}$ equals the intersection of all closed supersets of A.
3. Show that a subset of a metric space is bounded if and only if , it is nonempty and is contained is some closed sphere.

4. Prove the following properties on the boundary of a subset A of a metric space X:
   (i) the boundary of A equals $\bar{A} \cap \overline{A^1}$
   (ii) The boundary of A is a closed set.
   (iii) A is closed if, and only if, it contains its boundary.
5. Let X be a metric space. If {x} is a subset of X consisting of a single point, show that its compliment $\{x\}^1$ is open.
6. Let X be a metric space. If A is any finite subset of X, show that A^1 is open.
7. Let X be a metric space and $S_r\{x\}$, the open sphere in X with centre x and radius r. Let A be a subset of X with diameter less than r which intersects $S_r(x)$. Prove that A is contained $S_{2r}(x)$.
8. Let X be a metric space. Show that every subset of X is open if, and only if, each subset of X which consists of single point is open.
9. Let A and B be two subsets of a metric space X.

Prove the following:

(i) Int(A)∪ Int(B) ⊆ Int (A∪B)

(ii) Int(A)∩ Int(B) = Int(A∩B)

2.5 MODEL EXAMINATION QUESTIONS:

1. Define an open sphere. Show that the each open sphere is an open space in a metric space X and theϕ
2. Define an open set is a metric space X. Show that the empty set full space X are open sets in X.
3. Show that a subset G of a metric space X is open if, and only, if it is a union of open spheres.
4. Let X be a metric space. Show that any two distinct points of X can be seperated by open
   spheres.
5. Show that the subset [0,1) is not an open set in the set of real numbers R.
6. Let X be a metric space. Prove that
   (i) Any union of open sets is X is open, and
   (ii) Any finite intersection of open sets is X is open.
7. Prove that every non-empty open set on the real line is the union of countable disjoint class
   of open intervals.
8. Define(i) an interior point of a subset A of a metric space and (ii) the interior Int(A) of a subset A of X. Prove that A is open if, and only if ,A = Int(A).

2.6 SUMMARY:

An open set is a set where every point within it has a small neighborhood entirely contained within the set(meaning you can move a little bit around any point in the set and still stay inside0,while a closed set includes all its boundary points, essentially meaning any point that is close to the set must also be part of the set.

2.7 TECHNICAL TERMS:

- **Boundary Points**: The key difference lies in whether the set includes its boundary points-open sets do not, while closed sets do
- **Complement:** A set is closed if and only if its complement is open.

- **Open set:** The open interval(0,1) on the real number line, which includes all numbers between 0 and 1but not 0 and 1 themselves.
- **Closed set**: The closed interval on the real number line, which includes all numbers between 0 and 1, including 0 and 1.

2.8 SELF ASSESSMENT QUESTION:

1. In a metric space X, a subset A of X is open if, and only if A=Int(A).

Solution : Let X be a metric space and let A be a subset of X.

First, Let us assume that A is an open set implies that A is a neighbourhood of each of its points implies that every point of A is an interior point of A .Implies that A$\subseteq Int\ A ----1$ Also each interior point of A belongs to A. Implies that $Int\ A \subseteq A ----2.$

From (1) and (2), we have A=Int(A).

Conversly, Let us assume that A=Int (A) implies that A is an open set, since Int (A) is an open set.

Hence A is an open set if, and only if, A=Int(A).

2. The set of integers has no limit points.

Solution : Let n be any integers. For any real number 0<r<1,the open sphere (n-r, n+r) does not contain any integer except n. So n is not a limit point of the set $\{0, \pm 1, \pm 2, \dots\dots.. \pm n, \dots..\}$ of integers. This shows that the set of integers has no limit points.

3. The subset [0,1) is not an open set in R.

Solution : For any real number r>0 (however small r may be), the open sphere $s_r(0)$ = (-r, r) contains infinitely many points of (-r, 0), which do not belong to [0, 1], so that $s_r(0) \nsubseteq [0,1)$. So 0$\in$ [0,1) is not an interior point of [0, 1) and thus [0,1) is not an openset in R.

2.9 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON-3

CONVERGENCE AND COMPLETENESS IN METRIC SPACES - BAIRE'S THEOREM

OBJECTIVES :

- ❖ To introduce the concept of convergent sequence in metric spaces and to give its equivalent forms through open spheres and limit of a sequence.
- ❖ To introduce the concepts completeness and Cauchy sequences and illustrate them with suitable examples.
- ❖ To establish that every convergent sequence is Cauchy sequence but not vice-versa, by means of an example.
- ❖ To study the notions of limit and limit points of a sequence is a metric space and obtain the condition under which they are the same.
- ❖ To find a necessary and sufficient condition under which a subspace Y of a complete metric space X is complete.
- ❖ To prove Cantor's Intersection theorem on a decreasing sequence of subsets of metric spaces.
- ❖ To prove the two versions of Baire's Theorem.

STRUCTURE:

3.1 INTRODUCTION:

This lesson deals with the notions of the convergence, completeness and baire's theorems. The concept of limit and convergent sequences in the real number system which are studied in the real analysis can be successfully introduced in to metric spaces. The results resulting to limit of a sequence and convergence throw a greater insight to the corresponding notions on real number system and they complement each other. The Cantor's intersection Theorem and Baire's theorems relating sequences of sets in a metric space are also established in this lesson.

3.2 CONVERGENCE IN METRIC SPACE :

3.2.1 Definition : Let X be a metric space and Let $\{x_n\}= \{x_1,x_2,x_3,\ldots\ldots x_n,\ldots\ldots.\}$be a sequence in X. We say that $\{x_n\}$ is convergent in X, if there is a point x in X such that for each $\in > 0$, there exists a positive integer n_0 such that

$d(x_n,x) < \in$, for all $n \geq n_0$.

The points x_n which satisfy the condition $d(x_n,x) < \in$ lie in the open sphere $S_{\in}(x)$, centre x and radius $\in$, the above definition can equivalently be stated as: A sequence $\{x_n\}$ in a metric space X is convergent if, for each open sphere $S_{\in}(x)$,there exists a positive integer n_0 such that $x_n \in S_{\in}(x)$,for all $n \geq n_0$.

Since$\in$ >0 is arbitrary small, the statement that $\{x_n\}$ is convergent in X equally well defined as follows: the $\{x_n\}$ is convergent in X if, there exists a point $x \in X$ such that.$d(x_n,x) \to 0$ as $n \to \infty$.Symbolically, we write this be $x_n \to x$ as $n \to \infty$,and verbally express it by saying that x_n approaches x, or, x_n converges to x as $n \to \infty$ .The point x is called the limit of the sequence $\{x_n\}$ and we sometimes $x_n \to x$.

$\lim_{n\to\infty} x_n = x$, or simply $\lim x_n=x$

The statements $x_n \to x$ and $\lim x_n=x$ mean exactly the same, namely that $\{x_n\}$ is a convergent sequence with limit x.

3.2.2 : Theorem :

The limit of a convergent sequence in a metric space is unique.

Proof : Suppose x and y be two limits of a convergent sequence $\{x_n\}$.Then, given $\in$>0 and hence $\frac{\in}{2} > 0$, there exists positive integers n_1 and n_2 such that and

$$d(x_n,x) < \frac{\in}{2}, \text{ for all } n \geq n_1 \ldots\ldots\ldots(1)$$

$$d(x_n,y) < \frac{\in}{2}, \text{ for all } n \geq n_2 \ldots\ldots\ldots(2)$$

Let $n_0 = \max\{n_1,n_2\}$. Then (1) and (2) hold for all $n \geq n_0$.

Hence for all $n \geq n_0$.

$$\begin{aligned} d(x,y) &\leq d(x_n,x)+d(x_n ,y) \\ &< \frac{\in}{2} + \frac{\in}{2} \\ &= \in. \end{aligned}$$

Since $\in$ is arbitrary small, this gives x = y.

3.2.3. Theorem :

If the sequence $\{x_n\}$ is convergent, the given $\in > 0$, there exists a positive integer n_0 such that $d(x_m,x_n) < \in$, for all $m,n \geq n_0$.

Proof : Suppose that the sequence $\{x_n\}$ is convergent and convergence to x in X. Then given $\in > 0$ (and hence $\frac{\in}{2} > 0$),there exists a positive integer n_0 such that

$d(x_n,x) < \frac{\epsilon}{2}$, for all $n \geq n_0$……..(1)

So (1) holds for all $m,n \geq n_0$. Hence

$$d(x_m,x_n) \leq d(x_m,x)+d(x,x_n)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon \text{ for all } m,n \geq n_0.$$

3.3 COMPLETENESS IN METRIC SPACES :

3.3.1 Definition :

A sequence $\{x_n\}$ in a metric space (X,d) is called a Cauchy sequence, if given $\epsilon>0$, there exists a positive integer n_0 such that $d(x_m,x_n) < \epsilon$, for all $m,n \geq n_0$.

This means that all the terms of the sequence except finite number of them (possible after x_{n_0}) lie very close to each other.

3.3.2 Remark :

Theorem 3.3.3 shows that every convergent sequence in a metric space is a Cauchy sequence. However the converse need not be true. There do exist Cauchy sequences in a metric space (X, d), which are not convergent in X. The following example illustrates this.

3.3.3 Example :

An example of a Cauchy sequence which is not convergent.

Solution :

Consider the metric space

X = (0,1] = {x/x is a real number such that $0 < x \leq 1$}

(X is a metric space with respect to the metric d defined by $d(x, y)= |x - y|$)

Consider the sequence $\left\{\frac{1}{n}\right\}$ in x.

Let $\epsilon > 0$. Choose a positive integer n_0 such that $n_0 > \left[\frac{2}{\epsilon}\right]$, where $[x]$ denotes the positive integer part of x, or, the largest integer $\leq$ x. Then $n_0 > \left[\frac{2}{\epsilon}\right]$, so that $\frac{1}{n_0} < \frac{\epsilon}{2}$.

Then for all $m,n \geq n_0$, we have $\frac{1}{m}, \frac{1}{n} \leq \frac{1}{n_0} < \frac{\epsilon}{2}$.

Hence for all $m,n \geq n_0$,

$$d(x_m,x_n) = |x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right|$$

$$\leq \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right| = \frac{1}{m} + \frac{1}{n} (\text{since } m , n \geq 0)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} - \epsilon$$

So $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence in x. But the limit of the sequence $\left\{\frac{1}{n}\right\}$,that is $\lim_{n\to\infty} \frac{1}{n} = 0$, and $0 \notin [0,1] = x$.

Hence the sequence is not convergent in X = [0,1].

3.3.4 : Definition :

A metric space (X, d) is said to be a complete metric space, if every Cauchy sequence in X converges in X.

3.3.5 : Example : From the example 3.4.3, if follows that the metric space (X,d) X, is not a where X =[0,1] and d is given by d(x,y) = |x–y|, for all x,y $\in$X, is not a complete metric space.

3.3.6 : Example :

Using the nested intervals theorem in the space R of real numbers, it is well established that every Cauchy sequence in R is convergent in R. So R is a complete metric space.

Using the fact that R is a complete metric space the metric space C of ,complex numbers under the Metric d is defined by $d(z_1,z_2)=|z_1 - z_2|$ for all $z_1,z_2 \in C$ is also a complete metric space.

Limits and Limit points of sequences in Metric spaces :

The terms Limit and Limit point are often a source of confusion. On the real line R, consider the constant sequence {1, 1, ...} is convergent with limit 1. But treating it as a set, it is the singleton set {1} and hence it has no Limit points. This shows that a sequence may have a limit point but cannot have a limit. The following theorem gives the conditions under which a convergent a convergent sequence to have a limit point.

3.3.7 : Theorem :

If a convergent sequence in a metric space has infinitely many distinct points, then the limit of the sequence is a limit point of the set of elements of the sequence.

Proof : Let (X, d) be a metric space and Let $\{x_n\}$ be a convergent sequence in X with limit x. If possible, assume that x is not a limit point of the set $\{x_n\}$. Then there exists an given sphere $S_r(x)$ centred at x, which contains no point of the set $\{x_n\}$.

But x being the limit of the sequence $\{x_n\}$ there exists a positive integer n_0 such that $d(x, x_n)<r$, for all $n \geq n_0$

That is, the point x_{n_0}, x_{n_0+1} ,......... all lie in the open sphere $s_r(x)$ since $s_r(x)$ does not contain any point of $\{x_n\}$, we must have $x = x_{n_0} = x_{n_0+1} = \cdots$ .... Hence the sequence $\{x_n\}$ reduces to a finite sequence $\{x_1, x_2, ... x_{n0}\}$. This is a contradiction to the hypothesis that the sequence $\{x_n\}$ contains infinitely many distinct points. So our assumption that x is not a limit point of the set $\{x_n\}$, is wrong. Hence x is also a limit point of then set $\{x_n\}$.

The next theorem says where a sub space of a complete metric space is complete.

3.3.8 : Theorem :

Let X be a complete metric space and let Y be a subspace of X. then Y is complete if, and only if, it is closed.

Proof :

Let (X, d) be a complete metric space and let Y be a subspace of X. Assume that Y is complete as a subspace of X. We shall show that Y is a closed subspace of X. For this one has to show that Y contains all its limit points.

Let y be a limit point of Y. then for every $\in > 0$, the open sphere $S_{\in}(y)$ contains a point, say, y^1 of Y. That is $y^1 \in S_{\in}(y)$.Taking $\in = \frac{1}{n}$, n= 1,2,3,……, corresponding to each $\frac{1}{n} > 0$,there exists a point, say y_n of Y such that $y_n \in S_{\frac{1}{n}}(y)$. Then $d(y_m,y) \leq \frac{1}{n}$, n= 1,2,3,……..

Consider the sequence $\{y_n\}$ in Y. Since $d(y_n,y) = \frac{1}{n} \to 0 \; as \; n \to \infty$ it follows that $y_n \to y$ as $n \to \infty$, or, $\{y_n\}$ is a convergent sequence in Y. So $\{y_n\}$ is a Cauchy sequence Y. (since every convergent sequence is a Cauchy sequence).

But Y is complete. So the Cauchy sequence $\{y_n\}$ converges in Y. Since $\lim_{n \to \infty} y_n = y$, this implies that $y \in Y$. That is, Y contains all its limit points, so that Y is closed.

conversly assume that Y is closed. We shall show that Y is complete. Consider a Cauchy sequence $\{y_n\}$ in Y. Since $\{y_n \subseteq Y \subseteq X\}$, it follows that $\{y_n\}$ is also a Cauchy sequence in X.

Since X is a complete metric space, and since $\{y_n\}$ is a Cauchy sequence in X, it follows that $\{y_n\}$ converges in X. That is, x is a limit point of the set $\{y_n\}$ and $\{y_n\} \subseteq Y$ .So x is also a limit point of Y. Since Y is closed $x \in Y$.

That is, the Cauchy sequence $\{y_n\}$ in Y converges in Y, so that Y is complete.

Cantor's Intersection Theorem

3.3.9 : Definition :

Let (X, d) be a metric space and F, a subset of X. The diameter d(F) of the set F is defined as :$d(F) = sup\{d(x,y)/x,y \in F\}$

3.3.10 : Definition :

Let X be a metric space. A sequence {An } of subsets of X is called a decreasing sequence. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \ldots\ldots \supseteq A_n \supseteq \cdots \ldots$

The following theorem called cantors Intersection Theorem gives conditions under which the intersection of decreasing sequence of subsets of a metric space is non-empty.

3.3.11 : Cantor's Intersection Theorem :

Let (X, d) be a complete metric space and let $\{E_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Then $F = \bigcap_{n=i}^{\infty} F_n$ contains exactly one point.

Proof : Let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of the complete metric space (X, d) such that $d(F_n) \to 0$.

Since each F_n is non-empty, there exists a point. $x_n \in F_n$.

Consider the sequence $\{x_n\}$ in (X, d). Let m, n be any two distinct positive integers. Then either m<n or n<m. For definiteness, let m<n. Then $x_m \in F_m$ and $x_n \in F_n$. since $\{F_n\}$ is a decreasing sequence of subsets of (X, d) and since m<n, we have $F_n \subseteq F_m$.

So $x_n \in F_m$ and thus both $x_m, x_n \in F_m$.

By the definition $d(F_m) = sup\{d(x, y)/x, y \in F_m\}$,

$x_m, x_n \in F_m \Longrightarrow d(x_m, x_n) \in \{d(x, y)/x, y \in F_m\}$

$\Longrightarrow d(x_m, x_n) \leq sup\{d(x, y)/x, y \in F_m\}$

$\Longrightarrow d(x_m, x_n) \leq d(F_m)$.

Since, $d(F_n) \to 0$ as $m \to \infty$ (and have $n \to \infty$), it follows $d(x_m, x_n) \to 0$ as $m,n \to 0$ This shows that $\{x_n\}$ is a Cauchy sequence in (X, d).

But (X, d) is a complete metric space, so that the Cauchy sequence $\{x_n\}$ converges to a point in X. So there exists $x \in X$ such that $\lim x_n = x$.

We now show that $x \in \bigcap_{n=1}^{\infty} F_n$,so that $\bigcap_{n=1}^{\infty} F_n$, is a non-empty is required.

Let n_0 be a fixed but arbitrary positive integer. Here two cases will arise. Case (i) : Let $\{x_n\}$ be a finite. Then for some fixed integer r.

$x_r = x_{r+1} = x_{r+2} = - - - -$

Since $\lim x_n = x$, it follows that

$x = x_r = x_{r+1} = x_{r+2} = - - - -$

If $r < n_0$,then

$x = x_r = x_{r+1} = - - - - = x_{n_0} = - - -$

So that

$x \in F_{n_0}$

If $r > n_0$,then $F_r \subseteq F_{n_0}$and $x = x_r \in F_r \Longrightarrow x \in F_{n_0}$

In either case $x \in F_{n_0}$.

Case (ii): Let $\{x_n\}$ contain infinitely many distinct points. Then x, which is the limit of the sequence $\{x_n\}$ is also a limit point of the set $\{x_n\}$. Hence it is a limit point of the sequence

$$\{x_{n_0}, x_{n_0+1}, x_{n_0+2}, - - -,\}$$

which is a subset of F_{n_0}(since $F_{n_0} \supseteq F_{n_0+1} \supseteq F_{n_0+2} \supseteq - - - -$) so x is also a limit point of F_{n_0}. since $F_{n_0}$is a closed subset of X, the limit point x of $F_{n_0}$is an F_{n_0}.

Hence $x \in F_{n_0}$, for all positive integers, or, $x \in \bigcap_{n=1}^{\infty} F_n$ and thus $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Uniqueness : suppose there are two points $x, y \in \bigcap_{n=1}^{\infty} F_n$, then $d(x,y) < d(F_n) \to 0$ as n→∞. so x=y.

Baire's Theorem :

Any one of the following equivalent theorem's is called the Baire's category theorem, or, simply the Baire theorem.

We need the following concepts in the proof of the theorem.

3.3.12 : Definition :

A subset A of a metric space (X, d) is said to be a nowhere dense if its closure has empty-interior.

That is, A is said to be nowhere dense, if its closure $\bar{A}$ has no interior points, that is, for every x $\in \bar{A}$ there is a neighborhood G of x, which is not completely contained in $\bar{A}$.

This can be put in the following equivalent forms.

A is nowhere dense $\Leftrightarrow$ $\bar{A}$ does not contain any non-empty open set. (This follows from the above discussion) ⇔ each non-empty open set has non-empty open subset disjoint from A ⇔each non-empty open set has a non-empty open subset disjoint from A ⇔ each non-empty open set contains are open sphere disjoint from A.

Using these ideas we prove the first form of Baire's theorem.

3.3.13 : Theorem : (Baire's Theorem First Version)

If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space (X, d), then there exists a point in X, which is not in any of the A_n 's.

Proof : The metric space(X, d) is open (since X and ϕ are open sets in X)Since X is open and A_1 is nowhere dense in X , There is an open sphere S_1 of radius less than 1, which is disjoint from A_1 .

Let F1 be the concentric closed sphere (with S_1), whose radius is one half that of S_1 (that is, $\frac{1}{2}$) and consider its interior Int(F_1)(which is an open set).Since A_2 is no where dence. Int(F_1) (which is an open set). Contains an open sphere S_2 of radius less than $\frac{1}{2}$,which is disjoint from A_2 .

Let F_2 be the concentric closed sphere whose radius is one half that of S_2 (that is, $\frac{1}{2^2}$) and consider its interior Int(F_2)(which is an open set).

Since A_3 is nowhere dense, Int(F_2)(being an open set) contains an open sphere S_3 of radius less than $\frac{1}{2^2}$, which is disjoint from A_3 .

Let F_3 be the concentric closed sphere whose radius is one half that of S_3 (that is, $\frac{1}{2^3}$) and consider its interior Int(F_3)(which is an open set).

Continuing this way, we get a decreasing sequence $\{F_n\}$ of non- empty closed subsets of X, such that $d(F_n) < \frac{2}{2^n} = \frac{1}{2^{n-1}} \to 0 \text{ as } n \to \infty$, (since radius of F_n is less than $\frac{1}{2^n}$)

Since the metric space (X,d) is complete, by Cantor's Intersection Theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point, say, x of X. This point x is in all F_n 's and hence in all S_n 's. Since each S_n is disjoint from the corresponding A_n , it follows that x is not in An ,for all n. Thus there is a point x in X, which is not in any of A_n 's.

The following is an equivalent and most commonly used form of the above theorem.

3.3.14 : Theorem (Baire's Theorem-second version)

If a complete metric space is a union of a sequence of its subsets, the closure of at least one set of the sequence must have non empty interior.

Proof :

Let (X, d) be a complete space and Let $\{A_n\}$ be a sequence of subsets of X such that $X = \bigcup_{n=1}^{\infty} A_n$.

If possible, assume that the closure of every A_n has empty interior. Then each An is a nowhere dense subset of X and thus $\{A_n\}$ is a sequence of nowhere dense subsets of the complete metric space X. By the first version of Baire's theorem, There is a point x in X, which is not in any of the A_n 's.

That is,

$x \notin A_n$,for all n

So $x \notin \bigcup_{n=1}^{\infty} A_n = X$. This is a contradiction to the fact that $x \in X$. Hence our assumption that every A_n has empty interior is wrong and thus at least one A_n has non-empty interior.

3.4 EXERCISE :

(1) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$

(2) Show that a Cauchy sequence is convergent⇔ it has a convergent subsequence.

3.5 : MODEL EXAMINATION QUESTION:

(1) Define a convergent sequence is a metric space. Show that the limit of a convergent sequence in a metric space is unique.

(2) Define a Cauchy sequence. Show that a convergent sequence in a metric space is a Cauchy sequence. Give a example to show that a Cauchy sequence in a metric space need not to be convergent.

(3) If a convergent sequence in a metric space has infinitely many district points, then show that the limit point of the sequence is a limit point of the set of elements of the sequence.

(4) Let X be a complete metric space and let Y be a subspace of X. Prove that Y is complete if, and only if, Y is closed.

(5) Let (X, d) be a complete metric space and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Prove that $\bigcap_{n-1}^{\infty} F_n$contains exactly one point.

(6) If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, then prove that there exists a point x in X which is not in any of the A_n 's.

(7) If a complete metric space is a union of a sequence of its subsets. Prove that the closure of at least one set of the sequence must have non-empty interior.

3.6 SUMMARY :

Convergence refers to a sequence getting arbitrarily close to a specific point as the sequence progresses, while completeness means that every Cauchy sequence within that space has a limit that also lies within the space. Essentially ensuring that there are no holes where are converging sequence might not have a point to converge to within the space itself, a complete metric space is one where all Cauchy sequences converge to a point in the space.

3.7 TECHNICAL TERMS :

- Convergent Sequence: A sequence {x,n} in a metric space (x,d) converges to a point x in X if for every $\in > 0$, there exists N such that d(x_n,x)$<\in$ for all n$\geq$N.
- Limit: The point x to which a sequence {x_n} converges.
- Complete Metric Space: A metric space (x,d) is complete if every Cauchy Sequence in X converges to a point in X.
- Compactness: A metric space (X,d) is compact if every sequence in X has a convergent subsequence.

3.8 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON - 4

CONTINUOUS MAPPINGS IN METRIC SPACES

OBJECTIVES :

- ❖ To introduce the concept of continuity in a metric space(X,d) through the distance function d and to give its equivalent representation through open spaces.
- ❖ To establish a theorem expressing continuity through convergence of sequences in metric spaces.
- ❖ To Prove result that expresses continuity by means of open sets in metric spaces.
- ❖ To introduce the notion of uniform continuity in metric spaces and exhibit the difference between the continuity and uniform continuity by means of examples.
- ❖ To prove a theorem on the extension of a uniformly continuous function f defined on a dense subspace A of a metric space X in to a complete metric space Y to a uniformly continuous function from X into Y.

STRUCTURE :

4.1 INTRODUCTION :

The motion of continuity can be successfully introduced into the metric spaces through open spheres. In this lesson the continuity is charecterized by means of open sets in metric spaces. The motion of uniform continuity is also discussed.

4.2 CONTINUOUS FUNCTION:

In the previous lesson we introduced the notion of convergence in metric spaces and studied various aspects relating to it. In this lesson we do the same for continuity.

4.2.1 Definition :

Let X and Y be metric spaces with metrics d_1 and d_2 respectively and let f be a mapping of X into Y, f is said to be continuous at a point x_0 in X if, for every $\in > 0$,there exists $\delta > 0$ such that $d_1(x,x_0) < \delta$,implies that $d_2(f(x),f(x_0)) < \in$,

Since $d_1(x,x_0) < \delta \Leftrightarrow x \in S_\delta(x_0)$

and $d_2(f(x),f(x_0)) < \in \Leftrightarrow f(x) \in S_{\in}(f(x_0))$

the definition of continuity can be stated in the following equivalent form: f is said to be continuous at x_0 in X if, for every open sphere $S_{\in}(f(x_0))$centered on $f(x_0)$, there exists an open sphere S_δ (x_0) centered on x_0 such that

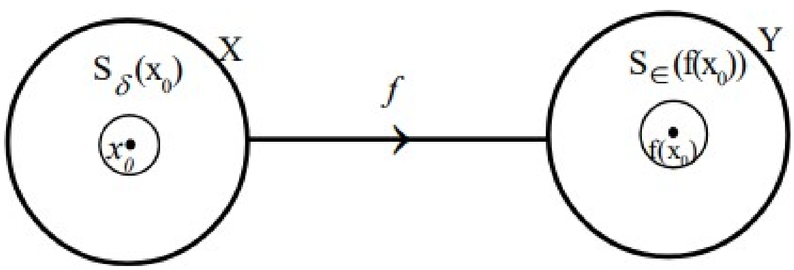
$$f(S_\delta(x_0)) \subseteq S_{\in}(f(x_0))$$

The following theorem expresses continuity at a point in terms of sequences which converge to that point.

4.2.2 Theorem :

Let X and Y be metric spaces and f:X $\rightarrow$ Y be a mapping. Then f is continuous at x_0 if, and only if, $X_n \rightarrow X_0 \Longrightarrow f(X_n) \rightarrow f(x_0)$.

Proof : Let (X,d_1) and (Y,d_2) be metric be spaces be metric spaces and let f:X$\rightarrow$ Y be a mapping. First let us assume that f is continuous at $x_0 \in$ X.



Let $\{x_n\}$ be a sequence in X such that $x_n \rightarrow x_0$ as $n \rightarrow \infty$.we have to show $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$.

Let $S_{\in}(f(x_0))$ be an open sphere centered on $f(x_0)$ in Y. Since f is continuous at x_0 , there exists an open sphere $S_\delta(x_0)$ in X such that

$$f(S_\delta(x_0)) \subseteq S_{\in}(f(x_0)) \text{............(1)}$$

Since $x_n \rightarrow x_0$ as $n \rightarrow \infty$, corresponding to $\delta > 0$, there exists a positive integer N such that $d_1(x_n,x_0) < \delta$ for all $n \geq N$, or, $x_n \in S_\delta(x_0)$, for all $n \geq N$.

So $f(x_n) \in f(S_\delta(x_0))$, for all $n \geq N$,or $f(x_0) \in S_{\in}(f(x_0))$, for all $n \geq N$,

Since $f(S_\delta(x_0)) \subseteq S_{\in}(f(x_0))$.

This shows that $d_2(f(x_n),f(x_0)) < \in$, for all $n \geq N$. That is $f(x_n) \rightarrow f(x_0)$ as $n \rightarrow \infty$.

Conversly, assume that $x_n \rightarrow x_0$ implies that $f(x_n) \rightarrow \infty\ f(x_0)$ as $n \rightarrow \infty$.

If possible, assume that f is not continuous at x_0 .Then there exists an $\in > 0$ such that $S_{\in}(f(x_0))$ does not contain that usage of any open sphere. $S_\delta(x_0)$ this is ,for every $\delta > 0$,this is an $x \in S_\delta(x_0)$ such that $f(x) \notin S_{\in}(f(x_0))$. Taking $= \frac{1}{n}$, n=1,2,3,……., there exists a sequence of points $\{x_n\}$ such that $x_n \in S_{\frac{1}{n}}(x_0)$ and such that $f(x_n) \notin S_{\in}(f(x_0))$.

Now $x_n \in S_{\frac{1}{n}}(x_0) \Rightarrow d_1(x_n,x_0) < \frac{1}{n} \to 0$ as $n\to\infty$.

That is $x_n \to x_0$ as $n \to\infty$ but $f(x_n) \notin S_{\epsilon}(f(x_0))$ for all n

$\Rightarrow d_2(f(x_n), f(x_0)) > \epsilon$

$\Rightarrow f(x_n)$ does not converge to $f(x_0)$.

This is a contradiction to the fact that $x_n \to x_0$ as $n \to\infty$ implies $f(x_n)$ as $n\to\infty$, $f(x_0)$ as $n\to\infty$ so our assumption that f is not continuous at x_0 is wrong and f must be continuous at x_0 is wrong and f must be continuous at x_0.

4.2.3 Definition :

Let X and Y be any two metric spaces A mapping $f:x\to Y$ is said to be continuous on X if, it is continuous at every point x of X.

4.2.4 Remark :

Continuity at a point x_0 of a metric space X is a local property, which is satisfied at x_0 only, where as continuity on X is a global property, that is a property satisfied at all points of X.

The following theorem is the consequence of the above theorem.

4.2.5 Theorem :

Let (X,d_1) and (Y,d_2) be metric spaces, and $f:X \to Y$ be a mapping Then f is continuous on X if, and only. $X_n\to X. \Rightarrow f(x_n) \to f(x)$, for all $x\in X$.

The next theorem charaecterized continuity in terms of open sets.

4.2.6 Theorem :

Let (X,d_1) and (Y,d_2) be metric spaces, and let f be a mapping X into Y. Then f is continuous if, and only, if $f^{-1}(G)$ is open in X whenever G is open in Y.

Proof : Let (X,d_1) and (Y,d_2) be metric spaces, and let $f:X\to Y$ be a mapping.

Assume that f is continuous. Let G be a open subset of Y.

Now $f^{-1}(G) = \{x\in X/f(x)\in G\}$.

If $f^{-1}(G)$ is empty, then trivially it is an open subset of X.

So let us assume that $f^{-1}(G)$ is non-empty. We shall show that every point of $f^{-1}(G)$ is an interior point of $f^{-1}(G)$, so that it is open subset of X.

Let $x\in f^{-1}(G)$ Then $f(x) \in G$. Since G is open in Y, f(x) is an interior point of G. So there exists an open sphere $S_{\epsilon}(f(x))$ such that

$$S_{\epsilon}(f(x))\subseteq G \quad \ldots\ldots\ldots..(1)$$

Since $f: X \to Y$ is a continuous mapping, corresponding to the open sphere $S_{\epsilon}(f(x))$ in G there exists an open sphere $S_{\delta}(x)$ in X centered on x, such that

$$f(S_{\delta}(x)) \subseteq S_{\epsilon}(f(x)) \subseteq G \text{ by,(1)}$$

So f($S_\delta(x)$) ⊆G,or, $S_\delta(x) \subseteq f^{-1}(G)$.That is, there exists an open sphere $S_\delta(x)$ centered on x such that $S_\delta(x) \subseteq f^{-1}(G)$. Hence x is an interior point of $f^{-1}(G)$, so that $f^{-1}(G)$ is open subset in X.

Conversely, assume that for every open subset G in Y, the set $f^{-1}(G)$ open in X. We shall show that f:X→ Y is continuous.

Let x∈X.Then f(x) ∈Y.Let be any open sphere centered on f(x) in Y. Then $S_\delta(x)$ is an open set in Y(since an open sphere is an open set).

By hypothesis $f^{-1}(S_\epsilon(f(x)))$ is an open set in X. Since f(x)∈ $S_\epsilon(f(x))$,it follows that x ∈ $f^{-1}(S_\epsilon(f(x)))$.

Since $f^{-1}(S_\epsilon(f(x)))$ is an open set in X. and since x ∈ $f^{-1}(S_\epsilon(f(x)))$, x is an interior point of $f^{-1}(S_\epsilon(f(x)))$. So there exists an open sphere $S_\delta(x)$ centered on x such that

$$S_\delta(x) \subseteq f^{-1}\left(S_\epsilon(f(x))\right).$$

This gives $f(S_\delta(x)) \subseteq S_\epsilon(f(x))$.

So f: X→Y is continuous at x, and hence an X.

4.3 UNIFORM CONVERGENCE:

Let (X,d_1) and (Y,d_2) be any two metric spaces. Suppose f: X→ Y is continuous at a point x∈ X. Then corresponding to $\epsilon > 0$,there exists a $\delta > 0$, such that whenever y∈X. and $d_1(x, y) < \delta$, we have $d_2(f(x),f(y)) < \epsilon$. Here δ depends on both ∈ and x. The following example explain this fact.

4.3.1 Example :

Consider the function f from the R→ R given by f(x) = 2x.

For any x ⊂ R and $\subset > 0$, consider, $\delta = \frac{\epsilon}{2}$ then for y ⊂ X, $|x \quad y| < \delta$ we have

$$|f(x) - f(y)| = |2x - 2y| = 2|x - y|$$
$$= 2\delta$$
$$< 2\frac{\epsilon}{2} = \epsilon.\text{So,}$$

the same $\delta\left(=\frac{\epsilon}{2}\right)$, serves for all x∈X and hence δ depends only on ∈.

4.3.2 Example :

Consider the function f:R→ R given by $f(x)=x^2$.

Since $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0}(a+h)^2$

$$= \lim_{h\to 0}(a^2 + 2ah + h^2) = a^2 + 2a.0 + 0^2$$

$= a^2 = f(a)$.

So, f is continuous at every point x $\in$ R.

However, we shall show that δ depends upon the point a also.

Let a $\in$ R and $\in$=2. Choose $\delta = \frac{1}{2}$.

For a=1 and x $\in$ R, such that $|x - y| < \frac{1}{2}$,

We have

$$= |f(x) - f(1)| = |x^2 - 1| = |x^2 - 1|$$

Now $|x - 1| < \frac{1}{2} \Leftrightarrow \frac{1}{2} < x < \frac{3}{2}$,

Since $x^2 - 1 = -\frac{3}{4}$ when x= $\frac{1}{2}$ and

$x^2 - 1 = \frac{5}{4}$ when x = $\frac{3}{4}$. So

$$|x - 1| < \frac{1}{2} \Leftrightarrow -\frac{3}{4} < x^2 - 1 < \frac{5}{4}$$

So, $|f(x) - f(1)| < 2(=\in)$.

But if a = 10, we have

f(x) – f(10) = $x^2 - 100$

So, if x = $10.\frac{1}{4}$, then $|x - a| = \left|10.\frac{1}{4} - 10\right| = \frac{1}{4} < \frac{1}{2}$.

But f(x) – f(10) = $x^2 - 10^2 = \left(10.\frac{1}{4}\right)^2 - 10^2$

$$= \left(10.\frac{1}{4} + 10\right)\left(10.\frac{1}{4} - 10\right)$$

$$= \frac{50}{4}.\frac{1}{4} - 3\frac{1}{8} > 2(-\in).$$

So, the same $\delta = \frac{1}{2}$ does not serve the purpose. Hence δ depends on $\in$ as well as the point a at which the continuity is discussed. We know consider a situation where δ depends only on $\in$ and is same for the all points in X.

4.3.3 Definition (Uniform continuity) ;

Let (X , d_1) and (Y , d_2) be metric space. A mapping f: X→ Y is said to be uniformly continuous on X, if given $\in > 0$ there exists a $\delta > 0$ such that for all x,y $\in$X and $d_1(x,y) < \delta \Longrightarrow d_2(f(x),f(y)) < \in$.

4.3.4 Definition :

A subset A of metric space (X , d) is called dense in X if $\bar{A}$= X, that is the closure of A in X.

4.3.5 Theorem :

Let (X,d_1) be a metric space and (Y,d_2) be a complete metric space and let A be a dense subspace of X. If f is uniformly continuous function of A into Y then f can be extended uniquely to a uniformly continuous mapping g of X into Y.

Proof : :Let (X,d_1) be a metric space and let A be a dense subspace of X.

Let (Y,d_2) be a complete metric space and let $f:A \rightarrow Y$ be a uniformly continuous function.

If A = X, then the conclusion is obviously true.

We there fore assume that $A \neq X$.

Define the function $g:X \rightarrow Y$ in the following way.

If X is the point of A, define g(x) = f(x)

If X is a point of x–A, define g(x) as follows:

Since A is dense in X, every point in X–A is a limit point of A, so that x is a limit point of A. Hence there exists a sequence $\{a_n\}$ in A which converges to x. Now $\{a_n\}$ is a convergent sequence, it is a Cauchy sequence in A.

Since $f:A \rightarrow Y$ is uniformly convergent, given $\in > 0$, there exists a $\delta > 0$ such that for all a,b $\in$A and $d_1(a,b) < \delta \Rightarrow d_2(f(a),f(b)) < \in$……..(1)

Again since $\{a_n\}$ is a Cauchy sequence is A, corresponding to $\delta > 0$, there exists a positive integer N, such that

$d_1(a_m,a_n) < \delta$, for all m,n $\geq$N ……..(2)

From 1&2, we have given, $\in > 0$, there exists a positive integer such that

$d_2(f(a_m),f(a_n)) < \in$,for all m,n $\geq$N.

This shows that $\{f(a_n)\}$is a Cauchy sequence in (Y,d_2).

Now Y being a complete metric space, the Cauchy sequence $\{f(a_n)\}$ in Y .converges to a point $y \in Y$.

That is,

$$\lim_{n\to\infty} f(S_n) = y$$

We shall now show that y depends only on x but not on the sequence $\{a_n\}$.

To see this let $\{b_n\}$ be other sequence in A such that $b_n \rightarrow x$ as $n \rightarrow \infty$.Then

$d_1(a_n,b_n) \leq d_1(a_n,x)+d_1(x,b_n)$

$\rightarrow 0+0$ as $n \rightarrow \infty$ [Since $a_n \rightarrow x$, $b_n \rightarrow x$ as $n \rightarrow \infty$]

So by the definition of uniform of $f : A \rightarrow Y$, it follows that (as in the above discussion)

$d_1(f(a_n),f(b_n)) \rightarrow 0$ as $n \rightarrow \infty$.

This gives, (Since $f(a_n) \rightarrow Y$ as $n \rightarrow \infty$),

$d_1((y,f(b_n)) \rightarrow 0$ as $n \rightarrow \infty$.

or, $f(b_n) \rightarrow y$ as $n \rightarrow \infty$.

This shows the limit y depends upon x.

Thus, for $x \in X$–A, we now define g(x) = y,

where y = limf(an) and {an} in the sequence in A such that $a_n \rightarrow x$ as $n \rightarrow \infty$

In this way g is defined for all x in X.

We next show that g is uniformly continuous.

Let $\in > 0$

Since $f:A \rightarrow Y$ is uniformly continuous corresponding to 0, there exists a $\delta > 0$ such that for a and a^1 in A we have.

$d_1(a,a^1) < \delta \Longrightarrow d_2(f(a),f(a^1)) < \in$............(3)

Let x and x^1 be any point on X such that $d_1(x,x^1) < \delta$. It suffices to show that $d_2(g(x),g(x^1)) < \in$.

By the definition of g,

$x = \lim a_n$ and $x^1 = a_n^1$,

for some sequences $\{a_n\}$ and $\{a_n^1\}$ in A.

By the triangle inequality we have

$d_1(a_n, a_n^1) \leq d_1(a_n, x) + d(x, x^1) + d_1(x^1, a_n^1)$.

Since $a_n \to x$, and $a_n^1 \to x^1$ as $n \to \infty$, and $d(x,x^1) < \delta$, it follows from the above inequality

$d_1(a_n, a_n^1) < \delta$ for all sufficiently large n. Hence, by (3), it follows that

$d_2(f(a_n),f(a_n^1)) < \in$................(4)

for all sufficiently large n.

$\Longrightarrow d_2(g(a_n),g(a_n^1)) < \in$, for sufficiently large n, since a_n, $a_n^1 \in A$.

$\Longrightarrow f(a_n) = g(a_n)$ and $f(a_n^1) = g(a_n^1)$.

Taking limit $n \to \infty$, this gives

$\text{Lim } d_2(g(a_n),g(a_n^1)) < \in$

Since $g(a_n) \to g(x)$ and $g(a_n^1) \to g(x^1)$, this gives

$d_2(g(x),g(x^1)) < \in$.

(Since $\{x_n\} \to x$ and $\{y_n\} \to y \Longrightarrow d(x_n,y_n) \to d(x,y)$).

That is for $x,x^1 \in X$. and $d_1(x,x^1) < \delta$

$d_2(g(x),g(x^1)) < \in$.

So $g:X \to Y$ is uniformly continuous.

Uniqueness :

We know that if $f: A \to Y$ and $g:X \to Y$ are continuous maps such that $f(a)=g(a)$ for every $a \in A$, then $f(a) = g(a)$ for every $a \in \bar{A}$, the closure of A.

In the theorem A=X. so if g and h are two uniformly continuous extensions of $f:A \to Y$, then $g(a) = h(a)$ for all $a \in A$, the $g(x) = h(x)$ for all $x \in \bar{A} = x$

So g= h on X and g is unique.

4.4 ISOMETRY:

4.4.1 Definition :

Let (X,d_1) and (Y,d_2) be any two metric spaces A mapping $f:X \to Y$ is called an isometry (or, an isometry mapping) if

$d_1(x,x^1) = d_2(f(x),f(x^1))$ for all x,x^1 X.

That is isometry between metric spaces is, distance preserving mapping.

Clearly isometry is one-one correspondence.

4.4.2 Example :

Isometry on metric spaces is a uniformly continuous map.

Solution : Let (X,d_1) and (Y,d_2) be any two metric spaces and $f:X\rightarrow Y$ be an isometry.

Let $\in >0$. Choose $\delta=\in$.

Then, for $x,y\in X$ and $d_1(x,y) < \delta(=\in)$, we have $d_2(f(x),f(y)) = d_1(x,y) < \delta=\in$.

(Since f is an isometry)

Since $\delta(=\in)$ depends on $\in$ only, f is uniformly continuous.

4.5 EXERCISES :

1. Let X and Y be metric spaces and let f be a mapping of X into Y. If f is a constant mapping, show that f is continuous.
2. Let X be a metric space with metric d, and x_0 be fixed point in X. Show that the real function f_{x_0} defined on X $f_{x_0}(x) = d(x,x_0)$ is continuous.
3. Let X and Y be metric spaces and A be a non-empty subset of X. If f and g are continuous mappings of X into Y such that f(x) = g(x) for every x in A, show that f(x) = g(x) for every x in $\bar{A}$.
4. Let X and Y be metric spaces and f a mapping of X into Y. Show that f is $\Leftrightarrow$continuous $f^{1}(F)$ is closed in X whenever F is closed in Y.

4.6 MODEL EXAMINATION QUESTIONS :

1. Define the concept (i) continuity at a point x of a metric space X and (ii) continuity in X.
2. Let X and Y be a metric spaces and let $f:X\rightarrow Y$ be a mapping. Show that f is continuous at $x_0\in X$ if, and only if, $x_n\rightarrow x_0 \Rightarrow f(x_n)\rightarrow f(x_0)$
3. Let $(X,d_1)$and (Y,d_2) Y be a mapping. Show$\rightarrow$) be metric spaces and let $f:X\rightarrow Y$ be a mapping. Show that f is continuous on X if, and only if, $x_n\rightarrow X\Rightarrow f(x_n)\rightarrow f(x_0)$, for all $x\in X$
4. Let $(X,d_1)$and (Y,d_2) be metric spaces and let $f:X\rightarrow Y$ be a mapping. Prove that f is continuous if, and only, if $f^{-1}(G)$ is open in X whenever G is open in Y.
5. Let X be a metric space and A be a dense sphere of X. Let Y be a complete metric space. If is a uniformly continuous function of A into Y, then prove that f can be extended uniquely to a uniformly continuous mapping g of X into Y.
6. Define isometry on metric spaces and show that it is a uniformly continuous mapping.
7. Let x be a metric space. Show that any two distinct points of X can be separated by open spheres.
8. Show that the subset[0,1) is not an open set in the set of real numbers R.
9. Let X be a metric space. Prove that (i) Any union of open sets in X is open, and (ii) Any finite intersection of open sets in the X is open.
10. Prove that every non-empty open set on the real line is the union of countable disjoint class of open intervals.
11. Define(i) an interior point of an subset A of a metric space and (ii) the interior Int(A) of a subset A of X. Prove that A is open if, and only if, A = Int(A).
12. Define a limit point of a subset A of a metric space X. Give an example to show that a limit point of a subset of a metric space need not to be point of the sub set.
13. If any metric space, show that the empty set and the full space X are closed sets.
14. Show that a subset F of a metric space X is closed if, and only if, its component F^1 is open.
15. Define a closed sphere in a metric space in a metric space. Show that a closed sphere in a metric space is a closed set.
16. Let X be a metric space. Prove the following

(i) Any intersection of closed sets in X is closed.
(ii) Union of finite number of closed sets in X is closed.

17. Define the closure of a subset A is a metric space X. show that A is closed if, and only if, A = $\bar{A}$.
18. Let X be metric space and let a be a subset of X. If x is a limit point of A, show that each open sphere centered on x contains an infinite number of distinct points of A.

4.7 SUMMARY :

A continuous mapping is a function that maps one topological space to another in a way that preserves certain properties. It is a generalization of the concept of a real valued continuous function.

4.8 TECHNICAL TERMS :

- Continuity on a space: A function f is considered continuous on the entire metric space X if it is continuous at every point in X.
- Open Set: A set U in a metric space (x,d) is open if for every x in U, there exists $\in > 0$ such that B(x,$\in$) $\subseteq U$.
- Relationship with sequences: A function is continuous if and only if for any sequence $\{x_n\}$ in X converging to x, the sequence $\{f(x_n)\}$ converges to f(x).

4.9 SELF ASSESSMENT QUESTIONS:

1.The continuous image of a compact metric space is …

a. Not compact
b. Compact
c. Disconnected
d. None of these

Ans: b

2. The function f: (0, 1) → $\mathbb{R}$ defined by $f(x)$ = 1 x is

a. Both continuous and Uniformly continuous
b. Uniformly continuous but not continuous
c. Continuous but not Uniformly continuous
d. Neither continuous nor Uniformly continuous

Ans: c

4.10 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON - 5

TOPOLOGICAL SPACES-DEFINITIONS AND SOME EXAMPLES

OBJECTIVE:

- A topological space is a set of points with a structure that defines how close points are
- This structure is called a topology and it's a collection of subsets of the set.
- The elements of the topology are called open sets

STRUCTURE :

5.0: INTRODUCTION:

The word topology, a branch of Mathematics, which is de-rived from Greek words has literal meaning, " the science of position". A topological property is a property of a " topological space" which is possessed by all topological spaces that are "homeomorphic" to the space. Topology can be defined as the study of all topological properties of topological spaces.

If we think of a topological space as a diagram drawn on a rubber sheet a homeomorphism may be thought of as any deformation of this diagram. A topological property, then would be any property of the diagram which is in- variant under any deformation. That is why topology is also called a rubber sheetgeometry.

In this lesson we start with the definition of a topology on a set, a topo-logical space, a subspace of a topological space, the topology generated by a class of subsets of a set and provide a good number of examples. A special typeof topological spaces , called metric spaces deserve separate attention because of their resemblance with the real line. We make a preliminary study of these spaces as well.

5.1: TOPOLOGICAL SPACES :

5.1.1: Definition : Let X be a non empty set. A class T of subsets of X is calleda topology on X if it satisfies the following conditions.

1. The union of every class of sets in T is in T. i.e. if $\{A_i / i \in I\}$ is any class

of sets in T indexed by a set 1. Then $\bigcup_{i=1} A_i$ O is in T and The intersection of every finite class of sets in T is in it i.e $C \subseteq T$ is anyfinite class of sets and G is the intersection of the sets in C then $G \in T$. If T is a topology on X we call the ordered pair (X, T) a topological space.

5.1.2: Remark : In definition 5.1.1 condition

(i)is described by saying that T is closed under arbitrary unions while condition is described by saying that T is closed under finite intersections.

5.1.3: Remark : The empty set is finite and the intersection of a family of subsets of X indexed by the empty set is the universal set X. Like wise the union of a family of sets indexed by the empty set is the empty set. Thus if (X, T) is a topological space, then $\phi \in T$ and $X \in T$.

5.1.4:SAQ : Show that condition (ii) of definition 5.1.1 holds if and only if $X \in T$, and $A \in T$, $B \in T \Rightarrow A \cap B \in T$

5.1.5: Example : Discrete Topology : Let X be a non empty set. For T we takethe power set P(X) of X,T is a topology. That is clear since P(X) contains all subsets of X and hence is closed under arbitrary unions and finite intersections. This topology is called the discrete topology of X.

5.1.6: Example : Let X be a nonempty set and T = {ϕ, X } , Clearly T is closed under arbitrary unions and finite intersections hence T is a topology on X. This topology is called the indisecrete topology on X.

5.1.7: Example : Suppose X is a nonempty set. We take T to be the class consisting of all $A \subseteq X$ where

i. Either $A = \phi$ or
X/A is a finite set

Then T is a topology on X. This topology on X is called the confinite topologyor the topology of finite complements.

Sol : Let $\{A_i / i \in I\}$ be any family of sets in T.
If $\bigcup_{i\in 1} A_i = \emptyset$ then $\bigcup_{i\in 1} A_i \in T$ by (i)

If $\bigcup_{i\in 1} A_i \neq \emptyset$ then $A_{i_0} \neq \emptyset$ for some i_0. Now

$X - \bigcup_{i\in 1} A_i = \bigcap_{i\in 1}(X/A_i) \subseteq X \backslash A_{i_0}$

Since $X\backslash A_{i_0}$, is finite,$\bigcap_{i\in 1}(X \backslash A_i)$ is finite.

$\therefore \bigcap_{i\in 1} A_i$ satisfies (ii) so lies in T
Hence T is closed under arbitrary unions $X\backslash X = \phi$, so $X \subset T$.
If A_1, A_2 are in T and $A_1 \cap A_2 \neq \phi$ then $A_1 \neq \phi \neq A_2$, so $X - A_1, X - A_2$ are finite. Hence $X - (A_1 \cap A_2) = (X - A_1) \cup (X - A_2)$ is finite. Hence $A_1 \cap A_2 \in T$.

Hence T is closed under finite intersection. Thus T is a topology on X.

5.1.8:SAQ: We fix a symbol ∞ which is different from every natural number and write $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The set T consists of all sets A where (i) $A \subseteq \mathbb{N}$ or (ii) $A \subseteq \mathbb{N}, \infty \in A$ and $\overline{\mathbb{N}} \setminus A$ is a finite subset of $\mathbb{N}$. Then T defines a topology on $\mathbb{N}$.

5.1.9: SAQ: If T_1, T_2 are topologies on X it is not necessarily true that $T_1 \square T_2$
Is a topology on X. Give an example.

5.1.10:SAQ: If T is a topology on X it is not necessarily true that T is closed under arbitrary intersections. Give an example.

5.1.11:Proposition: If $\{T_1 / i \in I\}$ is any class of topologies on a nonempty set X and $T = \bigcap_{i \in I} T_i$ then T is a topology on X. Further if T is any topology on X such that $T^1 \subseteq T_i$, $\forall i \in I$, then $T^1 \subseteq T$.

Proof: T is closed under arbitrary unions:$\{A_\alpha \backslash \alpha \in \Delta\} \subseteq T$

$\Rightarrow \{A_\alpha \backslash \alpha \in \Delta\} \subseteq T_i \forall i \in I$

$\Rightarrow A = \bigcup_{\alpha \in \Delta} A_\alpha \subseteq T_i \forall i \in I$ ($\therefore T_i$ is a topology)

$\Rightarrow A \in T_i \forall i \in I$

$\Rightarrow A \in T$

T is closed under finite intersections: If $C \subseteq T$ is any finite family of subsets of X, $C \subseteq T_i \forall i \in I$.

$\Rightarrow G = \bigcap_{A \in C} A \in T_i \forall i \in I$ (Since T_i is topology)

$\Rightarrow G \in \bigcap_{i \subset I} T_i$

$\Rightarrow G \in T$

Since T is closed under arbitrary unions and finite intersections T is a topology on X If T^1 is a topology on X such that $T^1 \subset T_i$,$\forall i \in I$, it is clear that $T^1 \subset T$.

Comparison of Topologies:

5.1.12: Definition: Let X be a non empty set and T_1, T_2 be topologies on X.We say that T_1 is weaker (= coarser) than T_2 and write in symbols $T_1 \leq T_2$ if$T_1 \subseteq T_2$. In this case we also say that T_2 is stronger (= finer) than $T_1$and write$T_2 \geq T_1$.

Remark : The indiscrete topology $T = \{\phi, X\}$ alone is contained in every topology on X so that T is weaker than every topology on X. Thus we may say that the indiscrete topology is the “weakest” topology on X.

Like wise the discrete topology P(X) consisting of all subsets of X is the "Strongest" topology on X as it is stronger than every topology on X.

5.1.13:Proposition: Let $A = \{A_\alpha \backslash \alpha \in \Delta\}$ be a collection of subsets of X. There is a unique topology T on X such that

1. $A \subseteq T$ and
2. $T \subseteq T^1$ for every topology T^1 containing A.

Proof: Let C be the class of all topologies T^1 on X containing A.

Since $P(X) \in C$, C nonempty if $T - \cap T^1, T^1 \in C$ by proposition 5.1.11 T is a topology on X.

Since $A \subseteq T^1, \forall T^1 \in C, A \subseteq T$.

If T^1 is any topology on X such that $A \subseteq T^1$ then $T^1 \in C$ so $T \subseteq T^1$

Thus $T = \bigcap_{T^1 \in C} T^1$ satisfies the required conditions.

If T_1 is a topology on X satisfying (i) and (ii),$T^1 \in C$ hence $T \subseteq T_1$

Since T satisfies (i) and (ii) $T \in C$ hence $T_1 \subseteq T$

Thus $T_1 = T$

This proves uniqueness.

5.1.14: Definition : Given $A = \{A_\alpha / \alpha \in \Delta\} \subseteq P(X)$, by the topology generated by A we mean the topology T which is the smallest topology containing A.

$T = \cap \{T_1 / T_1 \ topology \ on \ X, A \subseteq T_1\}$

T is also called the topology generated by A.

5.1.15: Proposition : Given any collection $\{T_\alpha / \alpha \in \Delta\}$ of topologies on X there is a unique topology T on X such that

(i) $T_\alpha \subseteq T \ \forall \alpha \in \Delta$

If T_1 is any topology such that $T_\alpha \subseteq T_1 \forall \alpha \in \Delta$,then $T \subseteq T_1$

Proof: Let C be the collection of all topologies on X that contain T_α for every $\alpha \in \Delta$ and T_0 be the intersection of all topologies in C.

$T_0 = \bigcap_{T^1 \in C} T^1$

By proposition 5.1.3 T_0 is the smallest topology on X containing $T_\alpha \forall \alpha \in \Delta$.

Thus T_0 is be required topology.

5.1.16:Theorem: Let X be a non-empty set and let T(X) be the class of all topologies on X. Let $\leq$ on T(X) be defined by $T_1 \leq T_2$ iff $T_1 \subseteq T_2$ for $T_1, T_2 \in T(X)$. Then $(T(X), \leq)$ in a complete lattice.

Proof: Clearly the indiscrete topology is the least element and the discrete topology is the greatest element in $(T(X), \leq)$.

Let $\{T_\alpha\}_{\alpha \in \Delta}$ be a non-empty family of topologies on X.

Let $T_1 = \bigcap_{\alpha \in \Delta} T_\alpha$

Let $A = \{T \in T(X) / T_1 \subseteq T \ \forall \alpha \in Y\}$

Let $T_2 = \bigcap_{T \in A} T$

Thus T_1 and T_2 are topologies on X and it is easy to verify that

$T_1 = g.l.b\{T_\alpha / \alpha \in \Delta\}$ and

$T_2 = l.u.b\{T_\alpha / \alpha \in \Delta\}$

Hence $(T(X), \leq)$ is a complete lattice.

5.1.17: proposition: Let T be a topology on a nonempty set X. $Y \subseteq X$ be a nonempty set and $T_Y = \{V \cap Y / V \in T\}$ Then T_Y is a topology Y.

Proof:(1) T_y is closed under arbitrary unions : Let $\{A_i / i \in I\}$ be an arbitrary class of sets in T_y. For each $i \in I, \exists B_i \in T \ni B_i \cap Y = A_i$

Hence $\bigcup_{i=I} A_i = \bigcup_{i=I}(B_i \cap Y) = (\bigcup_{i=I} B_i) \cap Y$, Since T is closed under arbitrary unions $\bigcup B_i \in T$ hence $\bigcup A_i \in T_Y$

(2) T_y is closed under finite intersections: Cleary $Y \in T_Y$. So it is enogh to prove that $A \in T_Y$ and $B \in T_Y \Rightarrow A \cap B \in T_Y$. Since $A \in T_Y \exists A_1 \in T_Y$ such that$A = A_1 \cap Y$

Similarly $\exists B_i \in T \ni B = B_1 \cap Y$.

Since $A_1 \in T$ and $B_1 \in T, A_1 \cap B_1 \in T$. Hence $A \cap B = (A_1 \cap B_1) \cap Y \in T_Y$

Hence T_Y is a topology on Y.

5.1.18: Definition : The topology $T_Y = \{A \cap Y / A \in T\}$ is called the relative topology on Y and (Y, T_Y) is called a subspace of (X,T). If (Y, T_y) is a subspace of (X,T) it is customary to say that Y is a subspace of X.

In definition 5.1.14 the topology generated by a given family A of subsets of a set $X(\neq \phi)$ is described as the smallest topology on X containing the given family A. In the following proble-m we provide a characterization for this topology.

5.1.19: Problem : Let X be a nonempty set and $A \subseteq P(X)$ write $T_1(A)$ for the family of subset of X each of which is the intersection of a finite class of sets in A and $T_2(A)$ for the family of subsets of X each of which is an arbitrary union of sets in A. Prove that $T_2(T_1(A))$ is the topology generated by A on X by providing the following.

1) $T_1(A)$ is closed under finite intersection and $A \subseteq T_1(A)$
2) $T_2(A)$ is closed under arbitrary unions and $A \subseteq T_2(A)$
3) $T = T_2(T_1(A))$ is a topology on X and $A \subseteq T_2(T_1(A))$
4) If T^1 is any topology on X containing A then T is contained in T^1

Solution: (1) Note that $T_2(T_1(A))$ is the family of all unions of finite intersections of set in A. If A is empty, then $T_1(A) = \{X\}$ and $T_2(T_1(A)) = \{\phi, X\}$. Clearly

$\{\phi, X\}$ is the topology generated by A. So, we may assume that A is nonempty**.**

Let $A = \bigcap_{C \in F_1} C \in T_1(A)$ and $B = \bigcap_{C \in F_1} C \in T_1(A)$ Where F_1, F_2 are finite subsets of A

Then $A \cap B = \bigcap_{C \in F_1 \cup F_2} C \in T_1(A)$ because $F_1 \cup F_2$ is a finite subset of A. Clearly $A \subseteq T_1(A)$.

(2) Let $\{A_\alpha / \alpha \in D\}$ be any arbitrary family of sets in $T_2(A)$. For each $\alpha \exists$ a set $F_\alpha \subseteq A \ni A_\alpha = \bigcap_{C \in F_\alpha} C$.

Then $F - \bigcup_{\alpha \in \Delta} F_\alpha \subseteq A$ and $\bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{\alpha \in \Delta}(\bigcup_{C \in F_\alpha} C) = \bigcup_{C \in F_\alpha} C \in T_2(A)$

Hence T_2 (A) is closed under arbitrary unions.

Clear$A \subseteq T_2(A)$.

(3) We claim that T = $T_2(T_1(A))$ is a topology on X containing A. Clearly T contains ϕ and X. From (2). T is closed under arbitrary unions. We show that T is closed under finite intersections. For this it is enough to show that A∈T, B∈T.

$\Rightarrow A \cap B \in T$

Let $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$ where $A_i \in T_1(A)$ and $B_j \in T_1(A)$ for $i \in I$ and $j \in J$.

Then$A \cap B = \bigcup_{i \in I, j \in J} C_{ij}$ where $C_{ij} = A_i \cap B_j$ since T_1 (A) is closed under finite intersections (by 1)

$A_i \cap B_j = C_{ij} \in T_1(A) \forall i \in I \text{ and } j \in J$

Then $A \cap B = \bigcup_{(i,j) \in I x J} C_{ij} \in T$

Since T is closed under finite intersections and arbitrary unions T is topology on X. Since $A \subseteq T_1(A) \subseteq T_2(T_1(A)$

(4). Let T^1 be any topology on X containing A. Since T is closed under arbitrary unions and finite intersections, $T_1(A) \subseteq T_1$ and hence T= $T_2(T_1(A)) \subseteq T_1$.

This completes the proof.

We now consider special type of topological spaces called metric spaces. A metric on a set X resembles the distance between real numbers and so several properties ofline distance on the real line $\mathbb{R}$ may be extended to a metric space.

5.1.20: Propositions: Let (X,d) be a metric space and T_d be the class of all open sets in X. Then T_d is a topology on X.

Proof: (1) Clearly ϕ and X are in T_d.

T_d closed under arbitrary unions: Let $\{G_i / i \in I\}$ be any class of sets in T_d and $G = \bigcup_{i \in I} G_i$, $x \in G \Rightarrow x \in G_i$ for some $i \in I$. Since G_i is open there exists $r > 0 \ni S_r(X) \subseteq G_i \subseteq G$ .Hence $S_r(X) \subseteq G$. Since this holds $\forall x \in G, G \in T_d$.

(2) T_d is closed under finite intersections: Let $G_1 \in T_d$ and $G_2 \in T_d$. $x \in G_1 \cap G_2$

$\Rightarrow X \in G_1$ and also $x \in G_2$.

$\Rightarrow \exists r_1 > 0 \text{ and } r_2 > 0 \ni S_{r_1}(x) \subseteq G_1 \text{ and } S_{r_2}(x) \subseteq G_2$

$\Rightarrow S_{r_1}(x) \cap S_{r_2}(x) \subseteq G_1 \cap G_2$

If $r = \min\{r_1, r_2\}$ *and* $y \in S_r(X)$ *then*

$$d(x,y) < r \Rightarrow d(x,y) < r_i \ (i = 1,2)$$

$$\Rightarrow y \in S_{r_i}(x) \quad (i = 1,2)$$

$$\Rightarrow y \in S_{r_1}(x) \cap S_{r_2}(x) \subseteq G_1 \cap G_2$$

$$\Rightarrow S_r(x) \subseteq G_1 \cap G_2$$

Since corresponding to every

$x \in G_1 \cap G_2 \exists\, r > 0 \ni S_r(x) \subseteq G_1 \cap G_2, G_1 \cap G_2 \in T_d$.

Hence T_d is closed under finite intersections. This shows that T_d is a topology on X.

5.1.21: Definition: The topology T_d on X defined in proposition 5.1.20 is called the topology on X induced by the metric d or simply the metric topology corresponding to d or the usual topology on the metric space X. The sets in T_d are called the open sets generated by the metric d on the space X.

5.1.22: Example:the usual topology on the Real line $\mathbb{R}$.

By an open interval in $\mathbb{R}$ we mean a set of the form (a,b)={x/x∈$\mathbb{R}$, a<x<b} where a∈$\mathbb{R}$, b ∈$\mathbb{R}$. A closed interval is of the form [a,b]={x/x∈$\mathbb{R}$, a≤ x ≤ b} and an open closed interval is defined to be (a, b]={x/x∈$\mathbb{R}$, a < x ≤ b}, and a closed. Open interval is defined to be [a, b) = { x i a≤ x≤ b}. The absolute value or modulus of x∈$\mathbb{R}$ is defined by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Let $T_U = \{G/G \subseteq \mathbb{R} \text{ and } \forall\, x \in G, \exists a\delta > 0 \text{ , such that } (x - \delta, x + \delta) \subseteq G\}$.

T_U is a topology on $\mathbb{R}$. This topology is called the usual topology on $\mathbb{R}$.Verification on the conditions for a topology.

(i) T_U is closed under arbitrary unions : Let $\{G_i / i \in I\}$ be an arbitrary class of sets in T_U and $G = \cup_{i \in I} G_i \,.\, x \in G \Rightarrow x \in G_i$ for some $i \in I$. Since for any such i, $G_i \in T_0 \ \exists\, \delta > 0 \ni (x - \delta, x + \delta) \subseteq G_i \Rightarrow (x - \delta, x + \delta) \subseteq G$.

This is true $\forall x \in G$. Hence $G \in T_U$

T_U is closed under finite intersections: Let $G_1 \in T_U$ *and* $G_2 \in T_U$

$$x \in G_1 \cap G_2 \Rightarrow x \in G_1 \text{ and } x \in G_2$$

$$\Rightarrow \exists\, \delta_1 > 0 \ni (x - \delta_1, x + \delta_1) \subseteq G_1 \text{and}$$

$$\delta_2 > 0 \ni (x - \delta_2, x + \delta_2) \subseteq G_2$$

$$\delta = \min\{\delta_1, \delta_2\}, (x - \delta\ x + \delta) \subseteq (x - \delta_1, x + \delta_1) \cap (x - \delta_2, x + \delta_2) \subseteq G_1 \cap G_2$$

Thus $x \in G_1 \cap G_2 \Rightarrow \exists\ \delta > 0 \ni (x-\delta, x+\delta) \subseteq G_1 \cap G_2$

Thus T_U is a topology on $\mathbb{R}$

Consider the real line $\mathbb{R}$. We define a metric on R by d(x,y) = |x–y|. This is called usual metric on $\mathbb{R}$. Note that the usual topology on $\mathbb{R}$ mentioned above is the same as the topology induced by d.

5.1.23: Note:

The Euclidean $\mathbb{R}^n$:

If n is a positive integer, $\mathbb{R}^n$ stands for the set of all n-tuples $(x_1, x_2 \ldots\ldots x_n)$ where $x_i \in \mathbb{R}$ for $1 \le i \le n$. If $n = 1$ *we write* $\mathbb{R}^1 = \mathbb{R}$ and identify (x_1) with x. n tuples x $=(x_1, x_2, \ldots.. x_n)$ and y $=(y_1, \ldots\ldots. y_n)$ are said to be equal if $x_i = y_i$ for $1 \le i \le n$. We define x + y = $(x_1 + y_1, \ldots\ldots\ldots\ldots, x_n + y_n)$. If $\alpha \in \mathbb{R}$. We define $\alpha x = (\alpha x_1, \ldots.. \alpha x_n)$. We define $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ and call this the norm of the vector x.

This norm is called the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{R}^n$ with this norm is called the Euclidean space.

5.1.24: Def (Euclidean Topology): The metric d defined by d(x,y) = ||x–y|| is called the Euclidean metric. The topology induced by the Euclidean metric is called the Euclidean topology on $\mathbb{R}^n$

5.1.25: The Unitary space $\mathbb{C}$: If n is a positive integer, $\mathbb{C}^n$ stands for the set of all n-tuples (z1…….zn) where $z_i \in \mathbb{C}\ \forall i,\ 1 \le i \le n$. If x = 1 we write $\mathbb{C}^1 = \mathbb{C}$ and identify (z) with z. If $z = (z_1 \ldots\ldots z_n)$ and $z^1 = (z_1^1, \ldots\ldots z_n^1) \in \mathbb{C}^n$. We say that $z = z^1$ when $z_i = z_i^1$ for $1 \le i \le n$.
We define $z + z^1 = (z_1 + z_i^1, \ldots.. z_n + z_n^1)$ and for $\alpha \in \mathbb{C}, \alpha z = (\alpha z_1, \ldots. \alpha z_n)$
For $z \in \mathbb{C}^n$ we define $\|z\| = \sqrt{|z_1|^2 + \cdots |z_n|^2}$ where z = $(z_1, \ldots\ldots\ldots. z_n)$
We call $\|z\|$, the norm of the vector z.

5.1.26: Spaces $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$:
We write K for either $\mathbb{R}$ or $\mathbb{C}$ and $K^{\mathbb{N}}$ for the collection of all sequences $\{x_n\}$ where $x_n \in K\ \forall n \in \mathbb{N}$ .We write K^∞ for all sequences $\{x_n\}$ in $K^{\mathbb{N}}$ for which $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. The space is $\mathbb{R}^\infty$ called the infinite dimensional Euclidean space while $\mathbb{C}^\infty$ is called the infinite dimensional Unitary space.

5.1.27: Example:
Let x be any nonempty set. The discrete metric d on X is defined by for x, y∈X

$$d(x,y) = \begin{cases} 0\ if\ x = y \\ 1\ if\ x \ne y \end{cases}$$

If $x \in X$ and $0 < r \le 1$, $S_r(x) = \{y \in X / d(x,y) < r < 1\} = \{x\}$
If $r > 1$ $S_r(x) = \{y \in X / d(x,y) < r\} = \{y / y \in X\} = X$
Consequently, if $G \subseteq X$ and $x \in G, S_1(x) \subseteq G$ this implies that the induced topology T consists of all subsets of X. Since T = P(X), the topology induced by the discrete metric is the discrete topology on X.

5.2: ANSWERS TO SAQ's:

SAQ: 5.1.4: If T is closed under finite intersections, $A \in T, B \in T$
$\Rightarrow A \cap B \in T$. As said in Remark 5.1.3 X ∈ T.
Conversely suppose $A \in T, B \in T \Rightarrow A \cap B \in T$ and also suppose X ∈ T
Then T contains the intersection of a family of sets in T indexed by the empty set.
If $\{A_1, \dots\dots\dots A_n\}$ is a non-empty finite family of sets in T
then $\bigcap_{i=1}^{n} A_i = \left[\bigcap_{i=1}^{(n-1)} A_i\right] \cap A_n$
Thus we can apply induction on n. If $n = 2$ $A_1 \cap A_2 \in T$ by hypothesis. Assume that $\bigcap_{i=1}^{n} A_i \in T$ whenever $A_i \in T$ for $1 \le i \le n-1$. Since $A_n \in T$ and $A = \bigcap_{i=1}^{n-1} A_i$, then $\bigcap_{i=1}^{n} A_i = A \cap A_n \in T$ by induction hypothesis. Hence the intersections of n elements in T is in T whenever it happens for $n-1$. So by induction this holds for all $n \in \mathbb{N}$.

SAQ :5.1.8:

Suppose $A_1 \in T$ *and* $A_2 \in T$. If one of A1, A2 is a subset of $\mathbb{N}$ then so is $A_1 \cap A_2$ so $A_1 \cap A_2 \in T$. If $\infty \in A_1 \cap A_2$ then $\mathbb{N}\backslash A_1$, $\mathbb{N}\backslash A_2$ are finite subsets of$\mathbb{N}$.Hence $\mathbb{N}\backslash A_1 \cap A_2 = (\mathbb{N}\backslash A_1) \cup (\mathbb{N}\backslash A_2)$ is a finite subset of $\mathbb{N}$. So $A_1 \cap A_2 \in T$. Thus T is closed under finite intersections as$\mathbb{N}$the intersections of an empty family lies in T by conditions (2) of this questions and by SAQ 5.1.4. That T is closed under arbitrary unions can be proved as in example 5.1.7.

SAQ: 5.1.9: X ={1,2,3}

$T_1 = \{\phi, \{1\}, X\} T_2 = \{\phi, \{2\}, X\}$

$T_1 \cup T_2 = \{\phi, \{1\}, \{2\}, X\}$

$\{1, 2\} = \{1\} \cup \{2\} \notin T_1 \cup T_2$

T_1 is a topology and T_2 is a topology but $T_1 \cup T_2$ is not a topology as $\{1, 2\} = \{1\} \cup \{2\} \notin T_1 \cup T_2$

SAQ: 5.1.10: Let $U_n = \{k / k \in \mathbb{N}, k \ge n\} \cup \{\infty\}$ for $n \ge 1$.
Here Un satisfies (2) of SAQ 5.1.8. So $U_n \in T, \forall n$
$\{\infty\} = \bigcap_{n=1}^{\infty} \notin T$.

5.3: MODEL EXAMINATION QUESTIONS:

1) Define a topology on a nonempty set X and a metric on X. Show that every metric induces a topology on X.
2) Let X be a nonempty set, $T = \{A / A \subseteq X, A = \phi \text{ or } X/A \text{ countable}\}$. Show that T is a topology on X.

3) Show that the class of topologies is a complete lattice with set inclusion.

4) If T_1, T_2 are topologies on a set X show that $T_1 \cup T_2$ is not necessarily a topology on X. Show also that there is a topology T on X containing both T_1 and T_2 and which is contained in every topology containing T_1, T_2.

EXERCISE:

1. Let X be a nonempty set and T be the class of all subsets of X whose complements are countable. Also let $\phi \in T$. Then show that T is a topology on X.
2. Let $Z \subseteq Y \subseteq X$. If T is a topology on X and T_y, T_z are the relative topologies on Y and Z respectively show that (Z, T_Z) is a subspace of (Y, T_Y) i.e. T_Z is the relative topology on Z with respect to the topology T_y on Y.
3. Let $X=\{a,b,c\}$ $T=\{\phi, \{a\}, \{a,b\} \{a,c\} X\}$. Show that {X, T) is a topological space.
4. Find all possible topologies on X when
   (a) $X=\{a\}$ (b) $X=\{a,b\}$ (c) $X=\{a,b,c\}$
   Compare the topologies obtained in
   (i) 4(a) (ii) 4(b) (iii) 4(c)
5. Show that the relative topology on the set Z of integers as a subspace of the real line with the usual topology is the discrete topology on Z.
6. Show that the indiscrete topology on a set consisting of at least two elements is not metrizable.
7. Let B be the collection of all open intervals (a,b) in $\mathbb{R}$, show that $T_2(B)$ is a topology on $\mathbb{R}$
8. Let B_0 be the collection of all intervals of the form (a,b) where $a \in Q$ *and* $b \in Q$ Show that $T_2(B_0)$ is a topology on $\mathbb{R}$.
9. Are the topologies $T_2(B)$,$T_2(B_0)$ in (7) and (8) equal ? Is any one of them equal to the usual topology?

5.4 SUMMARY :

Topological Spaces are Mathematical structures that define abstract relations of closeness and connectedness between objects in terms of relationships between sets rather than geometrical properties.

5.5 TECHNICAL TERMS :

Topological Space: A set X together with a collection τ of subsets of X, satisfying certain axioms, is called a topological spaces.
Topology: The collection τ of subsets of X is called a topology on X.
Hausdroff Space: A topological space X is said to be Hausdroff if for any two distinct points x,y in X, there exist neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$

5.6 SELF ASSESSMENT QUESTIONS:

5.1.4:SAQ : Show that condition (ii) of definition 5.1.1 holds if and only if $X \in T$, and $A \in T$

$B \in T \Rightarrow A \cap B \in T$

Answer: If T is closed under finite intersections, $A \in T, B \in T$
$\Rightarrow A \cap B \in T$. As said in Remark 5.1.3 X ∈ T.

Conversely suppose $A \in T, B \in T \Rightarrow A \cap B \in T$ and also suppose X ∈ T

Then T contains the intersection of a family of sets in T indexed by the empty set.

If $\{A_1, \ldots\ldots\ldots A_n\}$ is a non-empty finite family of sets in T

then $\bigcap_{i=1}^{n} A_i = \left[\bigcap_{i=1}^{(n-1)} A_i\right] \cap A_n$

Thus we can apply induction on n. If $n = 2$ $A_1 \cap A_2 \in T$ by hypothesis. Assume that $\bigcap_{i=1}^{n} A_i \in T$ whenever $A_i \in T$ for $1 \le i \le n-1$. Since $A_n \in T$ and $A = \bigcap_{i=1}^{n-1} A_i$, then $\bigcap_{i=1}^{n} A_i = A \cap A_n \in T$ by induction hypothesis. Hence the intersections of n elements in T is in T whenever it happens for $n-1$. So by induction this holds for all $n \in \mathbb{N}$.

5.1.9: SAQ: If T_1, T_2 are topologies on X it is not necessarily true that T_1□T_2
Is a topology on X. Give an example.

Answer: X ={1,2,3}

$T_1 = \{\phi, \{1\}, X\} T_2 = \{\phi, \{2\}, X\}$

$T_1 \cup T_2 = \{\phi, \{1\}, \{2\}, X\}$

$\{1, 2\} = \{1\} \cup \{2\} \notin T_1 \cup T_2$

T_1 is a topology and T_2 is a topology but $T_1 \cup T_2$ is not a topology as
$\{1, 2\} = \{1\} \cup \{2\} \notin T_1 \cup T_2$

5.1.10:SAQ: If T is a topology on X it is not necessarily true that T is closed under arbitrary intersections. Give an example.

Answer: Let $U_n = \{k / k \in \mathbb{N}, k \ge n\} \cup \{\infty\}$for $n \ge 1$.
Here Un satisfies (2) of SAQ 5.1.8. So $U_n \in T, \forall n$
$\{\infty\} = \bigcap_{n=1}^{\infty} \notin T$.

5.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON - 6

ELEMENTARY CONCEPTS IN TOPOLOGICAL SPACES

OBJECTIVES:

- Understand the basic definitions.
- Familiarize with topological properties.
- Develop understanding of topological operation.
- Apply topological concepts to simple spaces.
- Develop problem solving skills.
- Understand the relationship with other mathematical structures.

STRUCTURE :

6.0: INTRODUCTION:

The first countability axiom, the axiom of second countability are also introduced and the famous Lindelof's theorem along with simple but important consequences are proved. Separability in relation to second count-ability is also discussed.

6.1: CONCEPTS IN TOPOLOGICAL SPACES:

6.1.1:Definition: Let (X,τ) be a topological space . $V \subseteq X$ is said to be an open set or simply V is open in X or V is open if $V \in \tau$. $F \subseteq X$ is said to be a closed set or simply F is closed in X
or F is closed if its complement F=X/F is open in X. If $A \subseteq X$, the closure of A, denoted by $\bar{A}$ is the intersection of all closed supersets of A i.e., $\bar{A} = \cap \{F \backslash A \subseteq F, F \text{ is closed in } X\}$ A is dense in X if $\bar{A} = X$, In this case we simply say that A is every where dense or A is dense. (X,τ) is said to be a separable space or X is said to be separable if X has a countable dense subset.

6.1.2:Remark: Since the intersection of the empty family of sets (in τ) is the space X, $X \in \tau$. Since the union of the empty family of sets (in τ) is the empty set $\phi, \phi \in \tau$.Then ϕ and X are open sets in X consequently,ϕ and X are closed sets.

6.1.3: Examples : If X is a nonempty set, every subset of X is open, in the discrete topology and hence every subset of x is closed where as in the case of the indiscrete topology the only open sets are ϕ and X, hence the only closed sets are ϕ and X.

6.1.4:Propositions: The class Σ of all closed sets in a topological space (X, τ) has the following properties.

(i) $\phi \in \Sigma, X \in \Sigma$

(ii) $A \in \Sigma, B \in \Sigma \Rightarrow A \cup B \in \Sigma$

(iii) $\{A_i \backslash i \in I\} \subseteq \Sigma \Rightarrow \bigcap_{i \in I} A_i \in \Sigma$

Proof : (i) follows by remark 6.1.2

We use De Morgan's laws:

$(A \cup B)^1 = A^1 \cap B^1$ *and*

$(\bigcap_{i \in 1} A_1) \in \Sigma$

$If\ A \in \Sigma\ and\ B \in \Sigma \Rightarrow A^1 \in \tau, B^1 \in \tau$

$\Rightarrow A^1 \cap B^1 \in \tau$

$\Rightarrow (A \cup B)^1 \in \tau$

$\Rightarrow A \cup B \in \Sigma$

$A_i \in \Sigma\ \forall i \in I \Rightarrow A_i^1 \in \tau \forall i \in I$

$\Rightarrow \bigcup_{i \in I} A_i^I \in \tau$

$\Rightarrow (\bigcap_{i \in 1} A_i)^1 \in \tau$

$\Rightarrow \bigcap_{i \in 1} A_i \in \Sigma$

6.1.5: Corollary : The class Σ of all closed sets in a topological space is closed under finite unions and arbitrary intersections.

Proof follows from proposition 6.1.4.

6.1.6:SAQ: Suppose Σ is a class of subsets of a non-empty set X which is closed under finite unions and arbitrary intersections. Show that $Y = \{A^1 / A \in \Sigma\}$ is a topology on X.

6.1.7: SAQ: If (X, d) is a metric space $x \in X$ and $r > 0$, is $\overline{S_r(x)} = S_r(x)$? Justify your answer.

6.1.8: SAQ: In a metric space (X, d) show that $\{x\}$ is a closed $\forall x \in X$.

6.2: THE CLOSURE OPERATION:

We have defined the closure of a set A in a topological space (X, τ) to be the intersection of all closed sets containing A. The set X is closed as $\phi \in Y$ so that the collection of subsets of X that are closed in (X, τ) and containing A is nonempty. Moreover, the intersection of any class of closed sets is closed so that A is a closed set containing A. Moreover, $\bar{A}$ is the "Smallest" closed set containing A since every closed set F that contains A, also contains $\bar{A}$.

We will prove soon that this closure operation assigning $\bar{A}$ to an arbitrary set A in X satisfies "Kuratowski closure axions". We will also prove that any operation on P(X) satisfying these axioms induces a unique topology on X so that the closed sets in this topology are precisely those subsets of X that are in variant under this operation.

6.2.1: Proposition: Let (X,τ) be a topological space the operation $A \to \bar{A}$ from P(X) into P(X) where $\bar{A}$ is the closure of A satisfies the following:

$K_1: \bar{\phi} = \phi$

$K_2: A \subset \bar{A}$

$K_3: \bar{\bar{A}} = \bar{A}$ *and*

$K_4: \overline{A \cup B} = \bar{A} \cup \bar{B}$

Proof : By definition $\bar{A} = \cap\{F/F \text{ is closed and } F \supseteq A\}$. So $\bar{A} \subseteq F\ \forall$ closed $F \supseteq A$. In particular when $A = \phi, \bar{\phi} \subseteq \underline{\phi}$ since ϕ is closed. Then $\bar{\phi} = \phi$. This proves K_1.

If A ⊆ X , then A ⊆ ∩{F / A ⊆ F , F is closed}

Hence $A \subseteq \bar{A}$

If $A \subseteq X$, then $\bar{A}$ is a closed set and clearly $\bar{A} \subseteq \bar{A}$. Hence $\bar{\bar{A}} \subseteq \bar{A}$

Since $\bar{A} \subseteq \bar{\bar{A}}$ by K_2, it now follows that $\bar{\bar{A}} \subseteq \bar{A}$

Let $A \subseteq X$ and $\subseteq X$. Clearly by K_2, $\subseteq \bar{A}$,

$B \subseteq \bar{B}$ so that $A \cup B \subseteq \bar{A} \cup \bar{B}$. Since $\bar{A} \cup \bar{B}$ is closed.

$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Further $\overline{A \cup B} \supseteq A \cup B \supseteq A$

Since $\overline{A \cup B}$ is closed $\bar{A} \subseteq \overline{A \cup B}$. Similarly $\bar{B} \subseteq \overline{A \cup B}$

Therefore $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$

Hence $\overline{A \cup B} = \bar{A} \cup \bar{B}$. This process K_4

6.2.2:Theorem : Let X be a nonempty set. Suppose that with every subset A of X a set $\bar{A}$ is associated and that this association satisfies the following "Kuratowski closure axioms"

1) $\bar{\phi} = \phi$
2) $A \subseteq \bar{A}\ \forall A \subseteq X$
3) $\bar{\bar{A}} = A \forall A \subseteq X$ *and*
4) $\overline{A \cup B} = \bar{A} \cup \bar{B}\ \forall A \subseteq X \text{ and } B \subseteq X$

Then there is a unique topology $\bar{\tau}$ and X such that a set $A \subseteq X$ is closed in this topology if and only if $A = \bar{A}$.

Proof: Let $\Sigma = \{A/A \subseteq X \text{ and } A = \bar{A}\}$ clearly $\phi = \bar{\phi} \text{ and } X \subseteq \bar{X} \subseteq X$ so that $\bar{X} = X \text{ hence } \phi \in \Sigma \text{ and } X \in \Sigma$

$A \in \sum, B \in \sum \Rightarrow \bar{A} = A \text{ and } \bar{B} = B$

$\Rightarrow A \cup B = \bar{A} \cup \bar{B} = \overline{A \cup B}$

$\Rightarrow A \cup B \in \sum$

We prove that $\sum$ is closed under finite unions by using the principle of mathematical induction on the number of sets. Clearly this holds when n =1 and that this also holds for n = 2 is proved above. Now assume that the union ofany n sets in $\sum$ is in $\sum$.

Let $A_1, \ldots\ldots\ldots\ldots A_{n+1}$ set in Σ. Then i

$\overline{\cup_{i=1}^{n+1} A_i} = \overline{(\cup_{i=1}^{n} A_i)} \cup \overline{A_{n+1}}$

$= \cup_{i=1}^{n} A_i \cup A_{n+1} = \cup_{i=1}^{n+1} A_i$

Since $\overline{A_{n+1}} = A_{n-1}$, $\overline{(\cup_{i=1}^{n} A_i)} = \cup_{i=1}^{n} A_i$ by induction hypothesis. Hence $\cup_{i=1}^{n+1} A_i \,\varepsilon\Sigma$
This shows that, whenever $\sum$ is closed under union for n sets, $\sum$ is closedunder union for 3(n+1) sets.

Hence by induction $\sum$ is closed under finite unions.

We now show that $\sum$ is closed under arbitrary intersections . Let $\{A_i / i \in I\}$ be any nonempty class of sets in $\sum$. Then $\overline{A_i} = A_i \forall i \in I$
Clearly $\bigcap_{i\in I} A_i \subseteq \overline{\bigcap_{i\in I} A_i}$ (by 2). To prove the reverse inclusion we first note that $A \subseteq B$
$\Rightarrow A \cup B = B$
$\Rightarrow \overline{A \cup B} = \bar{B} \Rightarrow \bar{A} \cup \bar{B}$, $\bar{A} \subseteq \bar{B}$ (by 2)
Since $\bigcap_{i\in I} A_i \subseteq A_i \ \forall i \in I, \ \overline{\bigcap_{i\in n} A_i} \subseteq \overline{A_i} = A_i \forall i \in I$
Since this is true $\forall i \in I, \overline{\bigcap_{i\in n} A_i} \subseteq \bigcap_{i\in I} A_i$
This shows that $\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} A_i$, hence $\bigcap_{i\in I} A_i \in \Sigma$
Since $\bigcap_{i\in I} A_i \subseteq A_i \ \forall i \in I, \ \overline{\bigcap_{i\in n} A_i} \subseteq \overline{A_i} = A_i \forall i \in I$
Since this is true $\forall i \in I, \overline{\bigcap_{i\in n} A_i} \subseteq \bigcap_{i\in I} A_i$
This shows that $\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} A_i$, hence $\bigcap_{i\in I} A_i \in \Sigma$
Thus $\sum$ is closed under arbitrary intersections. By SAQ 6.1.6 , it followsthat Y={A/A$^1 \in \sum$} is a topology on X.

$A \subseteq X$ is closed in this topology.

$\Leftrightarrow A^1 \in Y \Leftrightarrow (A^1)^1 = A \in \Sigma \Leftrightarrow A = \bar{A}$

For uniqueness, suppose τ_0 is any topology on X such that A is closed in (X, τ_0) if $= \bar{A}$, for every subset A of X. Then A is closed in $(X, \tau_0) \Leftrightarrow A = \bar{A} \Leftrightarrow A$ is closed in (X, τ) $\therefore \tau = \tau_0$
This completes the proof of the theorem.

6.2.3: Corollary : Let Y be the unique topology on X obtained from the given operation $A \to \bar{A}$
from P(X) into itself as in the above theorem. Then for any $A \subseteq X$, the closure of A in (X, τ) is precisely $\bar{A}$.

Proof : For clarity, the closure of A in (X, τ) is denoted by A, for any subset A of X. Note that

$A = \cap \{F / F \text{ is closed and } F \supseteq A\}$

$\quad = \cap \{F / F \text{ and } F \supseteq A\}$

Since $\bar{A} = \bar{\bar{A}}$ and $\bar{A} \supseteq A$, $\bar{A} \subseteq \bar{A}$
But A is closed and $\bar{A} \supseteq A$
$\bar{A} \supseteq A \Rightarrow \overline{\overline{(A)}} \supseteq A \Rightarrow \bar{A} \supseteq \bar{A}$

Thus $\bar{A} = \bar{\bar{A}}$ as required

6.2.4: Definition : A neighborhood of a point x is a topological space (X, τ) is an open set containing x. A class $\sum$ neighborhood of a point x in a topological space (X,Y) is called an open base at the point x(or for the point x) if every neighborhood of x contains a member of

6.2.5:Example : If (X, d) is a metric space and x$\in$ X ,the class of open spheres $\{S_r(x) / r > 0\}$ is an open base at x because by definition every open set containing x contains $S_r(x)$ for some $r > 0$.

6.2.6: Proposition : Let (X, τ) be a topological space and $A \subseteq X$. Then $\bar{A} = \{x \,/\, x \in X$ and every neighbored of x intersects A $\}$.

Proof : Let B be the set specified on the right hand side. Let $x \in \bar{A}$ and V, any neighborhood of x. If $\cap A = \phi$, *then* $A \subseteq V^1$. Since V is an open set V^1 is a closed set containing A. Hence $\bar{A} \subseteq V^1$.
Since $x \in \bar{A}$, $x \in V^1$ so that $x \notin V$. This is a contradiction so that every neighborhood of intersects A , hence $\bar{A} \subseteq B$……………(1)
Now suppose that $x \in B$. We show that $x \in \bar{A}$, if $x \notin \bar{A}$ then $(\bar{A})^1$ is a neighborhood of x since $A \subseteq \bar{A}, A \cap (\bar{A})^1 = \phi$. This contradicts the assumption that every neighbourhood of x intersects A. Hence $x \in \bar{A}$ as required. There fore $B \subseteq \bar{A}$. This , together with (1) yields $\bar{A} = B$.

6.2.7: Definitions : Let X be a topological space $A \subseteq X$. A point $x \in A$ is calledan isolated point of A if it has a neighborhood V such that $V \cap A = \{x\}$ A point $x \in X$ is called a limit point of A
if each of a its neighborhoods contains a point of A other than x. The set of limit points of A is called the derived set of A and is denoted by D(A).

6.2.8: Remark : It is customary to call V\{x} a deleted neighborhood of x if Vis a neighborhood of x. Thus x is a limit point of A if and only if every deletedneighborhood of x intersects A .

A limit point of a set A is not necessarily a point of a where as an isolatedpoint of A must necessarily belong to A.

6.2.9: Proposition : Let X be a topological space and A be a subset of X, then

i) $\bar{A} = A \cup D(A)$
ii) $\bar{A} \backslash D(A)$ is the set of isolated points of A.
iii) $D(A) \subseteq A$ if and only if A is closed.

Proof : (1) If $x \in \bar{A}$, then by proposition 6.2.6 every neighborhood of xintersects A so that if $x \notin A$ every neighborhood of x intersects A in a point other than x so that x is a limit point of A, hence $x \in D(A)$. Thus $\bar{A} \subseteq A \cup D(A)$.
On the other $x \in D(A) \Rightarrow$ every neighborhood of x intersects A in a point other than x so that $x \in \bar{A}$.
Hence $D(A) \subseteq \bar{A}$. Since $A \subseteq \bar{A}, A \cup D(A) \subseteq \bar{A}$.
It is now clear that $\bar{A} = A \cup D(A)$.

(1) If $x \in \bar{A}$ and $x \notin D(A)$ there is a nbd V of $x \ni V \backslash \{x\} \cap A = \phi$ so that by (1) $V \cap A = \{x\}$. Hence x is an isolated point of A.
Conversely if x is an isolated point of A, then $x \in$ A and there exists a neighborhood V of $x \ni V \cap A = \{x\}$ so that $x \notin D(A)$. This implies that $x \in A/D(A) \subseteq \bar{A} \backslash D(A)$. Thus $\bar{A} \backslash D(A)$ is the set of isolated points of A.
(2) Since $\bar{A} = A \cup D(A)$ and A is closed if and only if $A = \bar{A}$, if follows that A is closed if and only if $A = A \cup D(A)$ if and only if $D(A) \subseteq A$.
As a consequence we have the following theorem.

6.2.10: Theorem : Let X be a topological space. Then any closed subset of X isthe disjoint union of the set of its isolated points and the set of its limit points inthe sense that it contains these sets, they are disjoint and it is their union.

Proof : Let A be a closed subset of X and i(A) the set of isolated points of A.
Then $i(A) \subseteq A, D(A) \subseteq A$ and by proposition 6.2.9. $i(A) = \bar{A}\backslash D(A) = A\backslash D(A)$ so that $i(A) \cap D(A) = \phi$ and $i(A) \cup D(A) = A$.

6.2.11: SAQ: Let X be a non empty set and T={ ϕ, X } be the indiscrete topology on X. Determine $\overline{\{x_0\}}$ for $x_0 \in X$.

6.2.12: SAQ: Let $\overline{IN} = N \cup \{\infty\}$ and T be the topology on $\overline{IN}$ described in problem 25 of lesson 2. Determine D(A) and $\bar{A}$ for $A \subseteq \overline{IN}$.

6.2.13:SAQ: Show by an example that D(A) is not necessarily closed for asubset A of a topological space.

6.2.14: SAQ: Show that if (X,d) is a metric space and $A \subseteq X$ then D(A) is a closed subset of X.

6.2.15: Definitions : Let (X,T) be a topological space and A be a subset of X.The interior of A, denoted by int(A) is the union of all open sets contained in A. A point $x \in A$ is called an interior point of A if $x \in X$ int (A); i.e, $x \in V$ some open set $V \subseteq A$ equivalently some neighborhood of x is contained in A.

6.2.16: Remark : It is clear that x is a boundary point of A if and only if everyneighborhood of x intersects A as well as its complement A^1.

6.2.17: Theorem : Let X be a topological space. Then any closed subset of X isthe disjoint union of its interior and its boundary; in the sense that it is their union.
Proof : Let $A \subseteq X$ be a closed set and $\partial(A) = \bar{A} \cap \overline{A^1}$ the boundary of A. When A is closed $D(A) = \bar{A} \cap \overline{A^1} \subseteq A$, clearly $\text{int}(A) \subseteq A$. If $x \in \text{int}(A)$, then some neighborhood V of x is contained in A so that $V \cap A^1 = \phi$ and so $x \notin \partial(A)$.
On the other hand if $x \in \partial(A), x \in A$ and every neighborhood V of x intersects A^1 so that V $\not\subset$ A . This implies that $x \notin$int(A). Hence int$(A) \cap \partial(A) = \phi$.
Clearly $x \in A$ and x is not an interior point of A , every neighborhood V of x intersects A^1 and A so that $x \in \partial(A)$. Hence $A = int(A) \cap \partial(A)$.
This completes the proof.

6.3: SOLUTIONS TO SHORT ANSWER QUESTIONS :

SAQ: 6.1.6: Use De Morgan's law :

$(\cap_{i\in 1} V_i) = \cup_{i\in 1} V_i^1$ and $(\cup_{i\in 1} V_i)^1 = \cap_{i\in 1} V_i^1$

$V_i \in \tau \forall i \in I \Rightarrow V_i^I \in \Sigma \Rightarrow \cap_{i\in I} V_i^I \in \Sigma \Rightarrow (\cup_{i\in I} V_i)^1 \in \Sigma$

$\Rightarrow \cup_{i\in I} V_i \in \tau$

If I if finite and $V_i \in \tau \forall i \in I$; $V_i^I \in \Sigma V_i \in I$

$\Rightarrow \cap_{i\in I} V_i^I \in \Sigma$

$\Rightarrow (\cup_{i\in I} V_i)^I \in \Sigma \Rightarrow \cup_{i\in I} V_i \in \tau$

SAQ :6.1.7: Let X be a set with at least two points and d be the discrete metric on X if $x \in X$, then $S_1(x) = \{x\}$
Since the topology induced by this metric is the discrete topology every subset of X is open hence closed . Thus $\overline{S_1(x)} = \{x\}$
However $S_1[x] = \{y \in x / d(x,y) \leq 1\} = X$
Thus it is not necessarily true that in a metric space $\overline{S_r(x)} = S_r[x]$

SAQ :6.1.8 : Let (X,d) be a metric space and $x \in X$. If X = {x}, {x} is closed.Suppose $\{x\} \neq X$.
Then $X\backslash\{x\} \neq \phi \forall y \in X\backslash\{x\}, r = d(x,y) > 0$
We show that $S_{r/2}(y) \subseteq X - \{x\}$. This holds since $z \in S_{r/2}(y)$
$\Rightarrow d(y,z) < \frac{r}{2}$. Since $D(y,x) = r, z \neq x$ so $z \in X\backslash\{x\}$
Then $S_{r/2}(y) \subseteq X - \{x\}$. This shows that X\x is an open set. Hence $\{x\}$ is a closed set.

SAQ: 6.2.11: We consider (X, j), j = {ϕ, x}.
Suppose $x_0 \in X$. If $x \in X$ and $\neq x_0$, then x is the only neighborhood of x in X. We have
$\{x_0\} \cap (X \setminus \{x\}) = \{x_0\}$
Therefore x is a limit point of the set $\{x_0\}$.
We have $\{x_0\} \cap (X \setminus \{x\}) = \phi$
And so x_0 is not a limit point of the set $\{x_0\}.\overline{\{x_0\}} = X$

SAQ: 6.2.12: We consider $\overline{\mathbb{N}}$
Suppose $A \subseteq \mathbb{N}$ is a finite subset. Then D(A) = ϕ and hence $\bar{A}$ = A
Suppose A is infinite.
Consider a neighborhood V of ∞ . By definition $V^1 = \mathbb{N} \setminus V$ is finite; therefore there are at least two points x,y in A which are not in V^1. Thus x,y∈V.
At least one of them is different from ∞ . Thus A ∩ (V \ {∞}) ≠ ϕ .
So ∞ is a limit point of A
Let n ∈$\mathbb{N}$. Then {n} is a neighborhood of n.
we have A ∩ ({n} \ {n}) = ϕ .
So n is not a limit point of A
Hence D(A) = {∞} and $\bar{A}$ = ∪{∞}

SAQ: 6.2.13: We consider a set X with at least two elements with the topology
T ={ϕ, X}. In SAQ 6.2.11. We have seen that the set X\{x} is the set of limit points of {x}. Since X contains at least two elements X\{x} is not empty and is not equal to x. Therefore it is not a closed subset of X:
D({x}) is not a closed set.

SAQ: 6.2.14: Let z be a limit point of D(A). To show that z ∈ D (A) we have to show that z is a limit point of A. Let V be a neighborhood of z. Since z is a limit point of D(A), ∃ a y ∈ D (A) ∩V such that y ≠ z . Since y ∈V ∃ an r > 0∋ S_r (y) ⊆ V . Since d(y,z) >0 we may choose r ∋ 0 < r < d (x, z) . Since y ∈ D (A), ∃ an $x \in S_r(y) \cap A \ni y \neq x$. Since

$x \in S_r(y), 0 < d(x,y) < r < d(y,z)$ so that $x \neq z$. Also $x \in S_r(y) \subseteq V$ .Thus $x \in V \cap A$ and $x \neq z$. Since every neighborhood V of z contains $x \neq z \ni x \in A$, z is a limit point A. There fore D(A) is a closed subset of X.

6.3 SUMMARY:

The concept of topological space is only one link in the chain of abstract space constructions which forms an indispensable part of all modern geometric thought. All of these constructions are based on a common conception of space which amounts to considering one or more systems of objects-points, lines, etc.

6.4 TECHNICAL TERMS:

- Topology: A collection of subsets of a set X, satisfying certain axioms.
- Open set : A subset U of X that belongs to the topology.
- Closed Set: A subset A of X whose complement X-A is open.
- Neighborhood: A set N containing an open set U with $x \in U$
- Interior : The largest open set contained in a set A, denoted by int(A)
- Closure: The smallest closed set containing a set A, denoted by cl(A)

6.5 SELF ASSESSMENT QUESTION:

6.1.6:SAQ: Suppose $\sum$ is a class of subsets of a non-empty set X which is closed under finite unions and arbitrary intersections. Show that $Y = \{A^1 / A \in \Sigma\}$ is a topology on X.

Answer: Use De Morgan's law :

$(\bigcap_{i\in 1} V_i) = \bigcup_{i\in 1} V_i^1$ and $(\bigcup_{i\in 1} V_i)^1 = \bigcap_{i\in 1} V_i^1$

$V_i \in \tau \forall i \in I \Rightarrow V_i^I \in \Sigma \Rightarrow \bigcap_{i\in I} V_i^I \in \Sigma \Rightarrow (\bigcup_{i\in I} V_i)^1 \in \Sigma$

$\Rightarrow \bigcup_{i\in I} V_i \in \tau$

If I if finite and $V_i \in \tau \forall i \in I$; $V_i^I \in \Sigma V_i \in I$

$\Rightarrow \bigcap_{i\in I} V_i^I \in \Sigma$

$\Rightarrow (\bigcup_{i\in I} V_i)^I \in \Sigma \Rightarrow \bigcup_{i\in I} V_i \in \tau$

6.1.7: SAQ: If (X, d) is a metric space $x \in X$ and $r > 0$, is $\overline{S_r(x)} = S_r(x)$? Justify your answer.

Answer: Let X be a set with at least two points and d be the discrete metric on X if $x \in X$, then $S_1(x) = \{x\}$

Since the topology induced by this metric is the discrete topology every subset of X is open hence closed . Thus $\overline{S_1(x)} = \{x\}$

However $S_1[x] = \{y \in x / d(x,y) \leq 1\} = X$

Thus it is not necessarily true that in a metric space $\overline{S_r(x)} = S_r[x]$

6.1.8: SAQ: In a metric space (X, d) show that{x} is a closed $\forall x \in X$.

Answer: Let (X,d) be a metric space and $x \in X$. If X = {x}, {x} is closed.Suppose $\{x\} \neq X$.
Then $X\backslash\{x\} \neq \phi \forall y \in X\backslash\{x\}, r = d(x, y) > 0$
We show that $S_{r/2}(y) \subseteq X - \{x\}$. This holds since $z \in S_{r/2}(y)$
$\Rightarrow d(y, z) < \frac{r}{2}$. Since $D(\text{y,x}) = r, z \neq \text{x so } z \in X\backslash\{x\}$
Then $S_{r/2}(y) \subseteq X - \{x\}$. This shows that X\x is an open set.

Hence $\{x\}$ is a closed set.

6.6 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Prof. L. Madhavi

LESSON -7

OPEN BASES AND OPEN SUB BASES

OBJECTIVES:

- Understand the definition of open bases.
- Understand the properties of open bases.
- Understand the definition of open subbases.
- Understand the properties of open subbases.
- Apply open bases and subbases to topological spaces.

STRUCTURE:

7.0: INTRODUCTION :

In the lesson , the first countability axiom, the second countability axiom are introduced and the famous lindelof's theorem along with simple but important consequences are proved. Separability in relation to second count-ability is also discussed.

7.1:OPEN BASES AND OPEN SUB BASES:

7.1.1:Definition: An open base for a topological space X is a class $\mathfrak{R}$ of opensubsets of X such that every open set in X is the union of class of sets in $\mathfrak{R}$. If $\mathfrak{R}$ is an open base for X sets of $\mathfrak{R}$ are called basic open sets.

7.1.2:Proposition: Let (X,T) be a topological space and $\mathfrak{R} \subseteq T$. $\mathfrak{R}$ is an open base for (X,T) if and only if $x \in G \subseteq Y \Rightarrow$ there exists a $B \in \mathfrak{R}$ such that $x \in B \subseteq G$.

Proof: Let $\mathfrak{R}$ be an open base, $x \in G \subseteq T$. By definition, there exists a class $\{B_i / i \in I\} \subseteq \mathfrak{R}$. Clearly $B_i \subseteq G$. Thus there exists $i \in I$ such that $x \in B_i \subseteq G$ and $B_i \in \mathfrak{R}$. Conversly suppose this condition is satisfied. Let $G \in T$ for each $x \in G$ there exists a $B_x \in \mathfrak{R} \ni x \in B_x \subseteq G$, $\{B_x / i \in G\} \subseteq \mathfrak{R}$ and clearly $G = \bigcup_{x \in G} B_x$. Hence $\mathfrak{R}$ is an open base for (X,T).

7.1.3:Remark: Let us recall that for any class of sets B, $T_2(B)$ is the class of sets that are unions of members of B. Thus we may rephrase the definition of anopen base as follows:
A class of open sets $\mathfrak{R}$ is a topological space (X,T) is an open base if and only if $T_2(\mathfrak{R}) = Y$.

7.1.4:Examples: Let X be a non empty set and T_d be the discrete topology on X. For each $x \in X$, let $B_x = \{x\}$. Then $\mathfrak{R} = \{B_i / x \in X\}$ is an open space for (X, T_d)

Reason: Let us recall that every subset of X is open in the discrete topology.

Thus if $G \subseteq X$, $G = \bigcup_{x \in G} B_x$.
Since $B_x \in \mathfrak{R}$ $\forall x \in G$, G is the union of a class of sets in $\mathfrak{R}$.
Hence $\mathfrak{R}$ is an open space for (X, T_d).

7.1.5:Example: For the real line R with the usual topology T_d the class B of all open intervals(a,b) where $a \in R$ and $b \in R$ is an open base.

Reason: By definition, G is an open set in the usual topology if and only if
$\forall x \in G\ \exists a\ \delta_x > 0 \ni$.
$I_x = (x - \delta_x , x + \delta_x) \subseteq G$.
Clearly $I_x \in B$ and $G = \bigcup_{x \in G} I_x$

7.1.6:Definition: Let (X,T) be a topological space. A class Y of open subset of X is said to be an open sub base for (x , y) if the class $B = T_1(y)$ is an open base for (X,T) where $T_1(y)$ confute of finite intersections of members of Y i.e, $A \in T_1(y)$ such that $A = \bigcap_{F \in Y} F$. Elements of Y are called sub basic open sets in (x,y).

7.1.7:Example: If $a \in R$ write $(-\infty, a) = \{x / x \in R \text{ and } x < a\}$ and $(a, \infty) = \{x / x \in R \text{ and } a < x\}$ $a < b$; The class $Y = \{(a,b) / a \in R,\ b = \infty \text{ or } a = -\infty,\ b \in R\}$ is an open sub base for the real line with the usual topology.

Reason: We know that the class $B = \{(a,b) / a \in R,\ b \in R\}$ is an open base for the space R with the usual topology.
If $a \in R$ and $b \in R$, $(a,b) = (-\infty, b) \cap (a, \infty) \in T_1(y)$
Further $(-\infty, a) = \bigcup_{x<a} (x, a) \in Y$ and $(b, \infty) = \bigcup_{b<x} (b, x) \in Y$
Thus $(-\infty, a)$ and (b, ∞) are the sub basic open sets
Hence $(a,b) \in T_1(y)$
Thus $B \subseteq T_1(y)$, since B is an open base, $T_1(y)$ is also an open base. Therefore y is an open sub base for R.

7.1.8:Theorem: Let X be any nonempty set and τ be an arbitrary class of subsets of X, then τ can serve an open sub base for a topology on X, in the sense that $T_2(T_1(\tau))$ is topology on X.

Proof: That $T_2(T_1(\tau))$ is the topology on X containing τ is proved in problem 5.1.19. By definition $T_2(T_1(\tau))$ is the collection of all sets which are arbitrary unions of members of $T_1(\tau)$ hence $T_1(\tau)$is the open base for this topology.
Thus τ is an open base for this topology.

7.1.9:Example: The collection $\mathfrak{R}_1$ of all open spheres is an open base for the Euclidean topology on R^2.

Reason: By definition $G \subseteq R^2$ is open in the Euclidean topology if $\forall x \in G$
$\exists$ an $r > 0 \ni x \in S_r(x) \subseteq G$. Hence $\mathfrak{R}_1$ is an open base for the Euclidean topology on R^2.

7.1.10:SAQ: The collection B_2 of all open rectangles is an open base and thecollection of all open strips is an open sub base for the Euclidean topology on R^2.

Countability axiom:

7.1.11:Definition: A topological space (X,T) is said to satisfy the first count- ability axiom or simply by first countable if every point in X has a countable open base (X,T) is said to satisfy the second countability axiom or simply be second countable if there is a countable open base for (X,T).

7.1.12:SAQ: Every metric space is first countable

Second countability axiom implies first countability axiom where as the reverse implication does not hold. Moreover a sub space of a first (or second) countable space is first (or second) countable. These two types of conditions play an important role in reducing the number of open sets in test cases.

We now prove the central fact about second countable spaces namely Lindelof's theorem and its consequence which is mostly used.

7.1.13:Lindelof's Theorem: Let X be a second countable space. If a non empty set G of X is represented as the union of class $\{G_i/i\in I\}$ of open sets then G can be represented as a countable union of G_is.

Proof: Let $\beta = \{B_n / n \in N\}$ be a countable open base for X.

Let $J_0 = \{n \in N / B_n \subseteq G_i\}$ for some $i \in I$ such that $B_n \subseteq G_i$ .Among all such i's we fix one and denote this by i_n, ie., $B_n \subseteq G_{i_n}$ Since J_0 is countable . Clearly $\{G_{i_n} / n \in J_0\}$ is a sub class of $\{G_i / i \in I\}$. We claim that $G = \bigcup_{n\in J_0} G_{i_n}$ clearly $\bigcup_{n\in J_0} G_{i_n} \subseteq G_i$.

Let $x \in G$, then $x \in G_i$ for some i. Since P is an open base, there exists an integer $n \geq 1$ such that $x \in B_n \subseteq G_i$ .Then $n \in J_0$ $x \in B_n \subseteq G_{i_n}$, by the choice of I_n.

Thus $x \in \bigcup_{n\in J_0} G_{i_n}$.

Hence $G \subseteq \bigcup_{n\in J_0} G_{i_n}$. There fore $G = \bigcup_{n\in J_0} G_{i_n}$

7.1.14:Theorem : Let X be a second countable space. Then any open base for X has a countable sub class which is also an open base.

Proof : Suppose (X ,τ) is a topological space which is second countable and we are given a basis $\{V_i/i\in I\}$ for τ, indexed by a set I. We show that there is acountable subset I_0 of I such that $\{V_i/i\in I_0\}$is an open base for τ . Since (X ,τ) is second countable, there is a countable open base $\Re = \{B_n / n \in N\}$ for τ . For each $n \in N$ there is a countable subset I_n of I (by Lindelof's theorem) such that $B_n = \bigcup_{i\in I_n} V_i$.

Let $I_0 = \bigcup_{n\in N} I_n$. Then I_0 is a countable subset of I.

We show that V = $\{V_i / i \in I_0\}$ is an open base for τ. Let us recall that for any class $Y \subseteq P(X)$.$T_2(Y)$ stands for the class of all sets which are unions of members of Y. Since $\Re$ is an open base for τ , $T_2 = (\Re) = \tau$.

Since $B_n = \bigcup_{i\in I_n} V_i$, $B_n = T_2(V)$

Hence $Y = T_2(B) \subseteq T_2(V)$. Since $V \subseteq Y$ and Y is closed under arbitrary unions.

$T_2(V) \subseteq \tau$ hence $Y = T_2(V)$. Thus V is a countable sub class which is an open base for τ.

7.2: SEPARABILITY AND SECOND COUNTABILITY :

7.2.1: Proposition : A second countable topological space is separable.

Proof : Let (X, τ) be a topological space with a countable open base $\{B_n / n \in N\}$.
Let J = $\{n / B_n \neq \phi\}$.

For each $n \in J$. Choose x_n in B_n. The set H = $\{x_n / n \in J\}$ is clearly countable. If $x \in$ X and V is any open set in X containing x, then there exists a $n \in N \ni x \in B_n \subseteq V$. So that $x_n \in V$. Thus every neighborhood of x intersects H. Hence $\bar{H} = X$ i.e., H is dense in X. Thus X is separable.

7.2.2: Remark : In general separability does not imply second countability (see excercise7) For metric spaces these two notions are equivalent as is evi- dent from the following theorem.

7.2.3: Theorem : Every seperable metric space is second countable.

Proof : Let X be a seperable metric space with metric d and A be a countable dence set we may enumerate the elements of A as $(a_1, a_2, \ldots, a_n)$.
For a fixed n, let $\Re_n = \{S_r(a_n)/r \in Q, r > 0\}$, clearly $\Re_n$ is countable.
Hence $\Re = \bigcup_{n \geq 1} \Re_n$ is countable. We show that $\Re$ is an open base for (X,d).
Clearly elements of $\Re$ are open spheres and hence are open sets. If V is any open set and $x \in V$, $\exists$a $\delta > 0 \ni S_\delta(X) \subseteq V$. Since A is dense, $\exists$ an $a_n \in S_{\delta/3}(x)$
Choose $r \in Q$, r.0 $\ni \delta/3 < r < 2\delta/3$
Since $d(x, a_n) < \delta/3 < r$, $x \in S_r(a_n)$.
$y \in S_r(a_n) \Rightarrow d(y,a_n) < r$.
$\Rightarrow d(y, x) \subseteq d(y, a_n) + d(x, a_n) < r + \delta/3 + \delta/3 = \delta$
$\Rightarrow y \in S_\delta(x)$. Hence $S_r(a_n) \subseteq S_\delta(x)$.
Thus $x \in S_r(a_n) \subseteq S_\delta(x) \subseteq G$.
Since $S_r(a_n) \in \Re_n \subseteq \Re$, it follows that $\forall G \in J$ and $x \in G$, $\exists$ a $\Re \in \Re \ni x \in B \subseteq G$.
Hence B is a basis for (X,d)
Since B is countable, (X,d) is second countable.

7.2.4:Example: The Euclidean space R with the usual metric is seperable, hence second countable.

Proof: We use Archimedean principle which says that if $\alpha \in \Re$ and $\alpha > 0$ there exists a natural number n such that $n > \alpha$.
As a consequence given a $a \in R, b \in R, a < b$ there exists $x \in Q \ni a < x < b$.
From this it follows that if $x \in R$ and $\in > 0, \exists y \in Q \ni x - \in < y < x$ so that $(x - \in, x + \in)$ contains a point of Q other than x. If V is neighborhood of x. $\exists$ an $\in > 0 \ni (x - \in, x + \in) \subseteq V$. Since $(x - \in, x + \in)$ contains a $y \in Q - \{x\}$, $y \in V \cap Q - \{x\}$. Hence x is a limit point of Q.
Since this
is true for every $x \in R$, $R \subseteq \bar{Q} \subseteq$ R. Hence R = $\bar{Q}$. Thus R is separable.

7.2.5:Example : R^n with the Euclidean metric is second countable, hence separable.

Proof : We use the fact that if $A_1 \ldots\ldots\ldots A_n$ are Countable then so is $A_1 x \ldots\ldots\ldots x A_n$. Since Q is separable. $Q^n = Q_x Q_x \ldots\ldots\ldots x Q$ (n times) is countable. Let $x = (x_1 \ldots\ldots x_n) \in R^n$; and V be a neighbourhood of x.

Then $\exists \in > 0 \ni S_\in(x) \subseteq V$. choose $y_i \in Q \ni x_i - \frac{\in}{\sqrt{n}} < y_i < x_i + \frac{\in}{\sqrt{n}}$ and $x_i \neq y_i$

Then $y = (y_1 \ldots\ldots y_n) \in Q^n$ and $d(x,y) = \{\sum_{i=1}^{n} |x_i - y_i|^2\}^{\frac{1}{2}} < \{\sum_{i=1}^{n} \in^2/n\}^{\frac{1}{2}} = \in$. Thus $y \neq x$ and $y \in S_\in(x) \subseteq V$. Since every neighborhood of x contains a point of Q^n other than x, $x \in \overline{Q^n}$. This is true for every $x \in R^n$ so that $R^n \subseteq \overline{Q^n} \subseteq R^n$ hence $R^n = \overline{Q^n}$. Hence R^n is seperable.

7.3: SOLUTIONS TO SHORT ANSWER QUESTIONS :

7.1.10: To prove that B_2 is an open base we have to show that every open rectangle is an open set and for every open set G is R^2 and $x \in G \exists$ an open rectangle R such that $x \in R \subseteq G$. Towards this end it is enough to show that if

$R = (a,b) \times (c,d)$ and $x = (x_1, x_2) \in R$, $\exists a\ \delta > 0 \ni S_\delta(x) \subseteq R$ and if $r > 0$ and $y \in S_r(x) \exists$ a rectangle $S = (\alpha, \beta) \times (\gamma, \delta) \ni y \in S \subseteq S_r(x)$.

Let $x = (x_1, x_2) \in (a, b) \times (c, d) \Rightarrow a < x_1 < b$ and $c < x_2 < d$

$\delta = \frac{1}{2}\min\{x_1 - a, b - x_1, x_2 - c, d - x_2\}, Y = (y_1, y_2) \in S_\delta(x)$

$\Rightarrow d(x,y) < \delta \Rightarrow (x_1 - y_1)^2 + (x_2 - y_2)^2 < \delta^2$

$\Rightarrow |x_1 - y_1| < \delta\ \&\ |x_2 - y_2| < \delta$

$\Rightarrow y_1 \in (x_1 - \delta_1, x_1 + \delta) \Rightarrow a < x_1 - \delta < y_1 < x_1 + \delta < b$ similarly $c < y_2 < d$ that $y \in (a,b)$. Thus $S_\delta(x) \subseteq (a,b)x(c,d)$.

Again , if $r > 0$ and

$y = (y_1, y_2) \in S_r(x), d(x,y) < r$ if $\delta = r - d(x,y), S_\delta(y) \subseteq S_r(x)$. The above argument shows that $S_\delta(y)$ contains a rectangle $S = (a,b)x(c,d)$ containing y.

Hence $y \in S \subseteq S_r(x)$.

7.1.12: Let (X, d) be a metric space and $x \in X$. Then $\{S_{1/n}(x) / n \in \mathbb{N}\}$ is a countable collection of open sets which form an open base at x. For this let V be a neighborhood of x is the induced topology $\exists$ and $\varepsilon > 0$, such that $x \in S_\varepsilon(x) \subseteq V$.

If $n \in N$ and $n > \frac{1}{\in}$, then $\frac{1}{n} > \in, S_{1/n}(x) \subseteq S_\in(x) \subseteq V$. Thus every neighborhood of x contains $S_{1/n}(x)$ for some $n \in \mathbb{N}$. This completes the proof.

7.4: MODEL EXAMINATION QUESTIONS :

1. Define an open base for a topology τ. Show that given any nonempty family τ of subsets of a nonempty set X there is a unique topology τ on **** for which Y is an open sub base.
2. State and prove Lindelof's theorem.
3. Show that every open base of a second countable topological space contains a countable sub family which is a base.
4. Define first countable topological space and second countable topological space. Show that a second countable topological space is first countable butthe converse is not true. Show that in a second countable topological spaceevery open set is a union of a countable family of open sets.
5. State Kuratowski's closure axioms and prove that any closure operation"-" satisfying these axioms induces a topology τ on X such that for any subset A of X, A = A iff

$X \backslash A \in \tau$.

6. Show that in a topological space (X, τ), for any $A \subseteq X$ $\bar{A} = A \cup D(A) = \{x \in X /$ every neighbourhood of x intersects $A\}$

7. Is a metrizable space first countable ? Justify.

8. Show that in any topological space (X, τ), int(A) is the "Largest" open set contained in A more precisely.
   (1) Int(A) is an open set,
   (2) int(A) $\subseteq$ A. If B is any open set $\ni$ B ? A, Then $B \subseteq$ int(A)

9. Let X be a nonempty set and consider the class τ of subsets of X consisting of the empty set ϕ and all sets whose complements are countable.For definiteness let X = R, the real line.
   1) Is X first countable ?
   2) Is X second countable ?
   3) Find $\bar{A}$ when A is the set of even integers.

10. Let (X, τ) be a topological space, $A \subset X$. Show that A is dense in $\bar{A}$ when $\bar{A}$ is treated as a sub space of (X, τ).

11. Let (X, τ) be a topological space and $A \subseteq X$. Show that the following conditions are equivalent.
   1) A is closed and has no isolated points
   2) $\bar{A}$ = D(A)

$A \subset X$ is said to be perfect, if A satisfy one of the above two conditions.

12. Let C be the cantor set in [0,1] obtained by removing the middle onthird at every stage. Show that C is perfect

13. For any $A \subseteq (X, \tau)$ show that the int $(A^1) = (\bar{A})^1$

14. Show that $\bar{A} = A$ iff A contains it boundary.

15. Let (X, τ) be a topological space for $A \subset x$. Show that int $(\bar{A}) = \phi$ if and only if non-empty open subset disjoint from A. Such sets A are called no where dense sets.

16. (a) Show that a closed subset a of (X, τ) is no where dense iff A^1 is dense
   (b) Consider that the real line with the usual topology

   (i) Is Q dense ?
   (ii) Is Q now where dense
   (iii) Is Q Closed
   (iv) Is Q open ?

17. Show that the boundary of a closed set is nowhere dense. What is theboundary of Q in R with the usual topology ?

18. Show that the set of isolated points of a second countable space iseither empty or countable.

19.Show that (X, τ) is second countable and $Y \subseteq X$ is uncountable then $D(Y) \neq \phi$

20. Let (X, τ) be a topological space and $A \subseteq X$. Show that boundary of $A = \phi$ if and only if $A \in \tau$ and $A^1 \in \tau$

7.5 SUMMARY:

An open base is a collection of open sets within a topological space that can be used to construct all other open sets by taking unions of its elements. While an open subbase is a collection of open sets where taking finite intersections of its elements generates a base for the topology, essentially acting as a smaller building block for creating the full set of open sets

7.6 TECHNICAL TERMS :

- Base: A collection of open sets in a topological space that generates the topology.
- Open Base: A base that consists only of open sets.
- Base for a topology: A collection of open sets that generates the topology.
- Subbase: A collection of open sets in a topological space that generates a base.
- Standard Base: The collection of all open intervals in the real line is a base for the standard topology.

7.7 SELF ASSESSMENT QUESTIONS:

7.1.12:SAQ: Every metric space is first countable

Answer: Let (X, d) be a metric space and $x \in X$. Then $\{S_{1/n}(x) \,/\, n \in \mathbb{N}\}$is a countable collection of open sets which form an open base at x. For this let V be a neighborhood of x is the induced topology $\exists$ and $\varepsilon > 0$, such that $x \in S_\varepsilon(x) \subseteq V$.
If $n \in N$ and $n > \frac{1}{\epsilon}$, then $\frac{1}{n} > \epsilon, S_{1/n}(x) \subseteq S_\epsilon(x) \subseteq V$. Thus every neighborhood of x contains $S_{1/n}(x)$ for some $n \in \mathbb{N}$. This completes the proof.

7.8 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition .

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LESSON –8

WEAK TOPOLOGY

OBJECTIVES :

- ❖ The weak topology allows for more convergent sequences and compact sets than the norm topology.
- ❖ The weak topology is a powerful tool for studying these.
- ❖ The weak topology helps analyze convergence of sequences and series in infinite-dimensional spaces.
- ❖ The weak topology is useful in applications to optimization, quantum mechanics, and differential equations.

STRUCTURE:

8.0: INTRODUCTION :

In this lesson, we first define partial ordered set, lattice, complete lattice and weak topology and prove the theorem that the collection of all topologies on X forms a complete lattice under the relation is weaker than. We then turn to our attention to prove properties of continuous functions real or complex functions defined on a topological space X.

8.1: CONTINUITY :

Definition : Let X,Y be topological spaces. A mapping f: X → Y is said to be continuous at $x_0 \in X$ if for every neighborhood V of $f(x_0)$ there is a neighborhood U of x_0 .such that $f(U) \leq V$.

8.1.2 : Example : Let X be any nonempty set. Equip X with the discrete topology. If Y is any topological space and f: X→Y is any map and x_0 is any point of X, f is continuous at x_0 because every subset of X is open with respect to the discrete topology and in particular for every open set V containing $f(x_0)$, $U=f^{-1}(V) \subseteq X$ is an open set.

8.1.3 : Note: Let X,Y be topological spaces and f: X→Y be any map. f is said to be Continuous at every point of X if and only iff^{-1}(V) is open in X for every open set V in Y.

8.1.4 : Definition : Let X and Y be topological spaces and f:X→Y be a mapping.

1) f is said to be continuous if $f^{-1}(G)$ is open in X, ∀ open sets G in Y.
2) f is said to be open if f(G) is open in Y ∀ open sets G in X.
3) f is said to be homeomorphism, if f is a bijection and is both continuous and open.
4) f(X) is said to be a continuous image of X if f is continuous.
5) f(X) is said to be a homeomorphic image of X if f : X→ Y is continuous, one-one and open.

8.1.5 : Remark : Some authors prefer to define continuity of f in terms of continuity at every point of X. However 8.1.3 confirms equivalence of these two definitions.

8.1.6 : Example : If X is a nonempty set and τ_i, τ_d are respectively the indiscrete topology and discrete topology on X respectively then the identity map I: X→ X is clearly a bijection.

When the domain space X is equipped with the discrete topology every set in X is open in τ_d . where as the only open sets in τ_i are ϕ and X. Thus

a)If X has more than one point I: (X, τ_i)→(X, τ_d) is not continuous.

b)As mentioned in 8.4.1 : (X, τ_d)→(X, τ_i) is continuous.

c)If X has more than one element and $x \in X, I(\{x\}) = \{x\} \neq X$ and is nonempty so that I : (X, τ_d)→(X, τ_i) is not open, hence is not a homomorphism.

8.1.7 : Let (X,τ) and (Y,σ) be topological spaces. For mapping $f: X \longrightarrow Y$ prove that following
are equivalent.

a)f is continuous.

b)$f^{-1}(F)$ is closed in (X, τ) for every closed set F in (Y, σ)

c)$f(\bar{A}) \subseteq f(\bar{A}) \forall A \subseteq X$

8.1.8 :Theorem : Let X,Y be topological spaces.$\Re$ be, an open base for X and σ an open subspace for Y. Then the following are equivalent.

a)$f: X \longrightarrow Y$ is continuous.

b)f^{-1}(B) is open in X for every basic open set B in Y.

c) f^{-1}(B) is open in X for every B∈ σ

Proof : (a) ⟺ (b) is clear

(b) ⟺ (c) since $\sigma \subseteq T_1(\sigma)$ and $T_1(\sigma)$is an open base for γ

(c) ⟺ (a): Let V be open in Y, x ∈ X and =f(x) ∈ V.

Then since $T_1(\sigma)$ is an open base for the topology on Y, ∃B∈$T_1(\sigma)$ such that Y∈B⊆ V . Since B∈ $T_1(\sigma)$, ∃ a finite number of sub basic open sets B_1……..B_n such that B = $\bigcap_{i=1}^{n} B_i$.

Since $y \in B, y \in B_i$ for $1 \leq i \leq n$ by (c) $f^{-1}(B_i)$ is open in X∀ i.

Hence $f^{-1}(\bigcap_{i=1}^{n} B_i) = \bigcap_{i=1}^{n} f^{-1}(B_i)$ is open X.

Since y = $f(x) \in B_i, x \in f^{-1}(B_i) \forall i$, so that $x \in \bigcap_{i=1}^{n} f^{1}(B_i) = f^{-1}(\bigcap_{i=1}^{n} B_i)$

Thus $x \in G = \bigcap_{I=1}^{n} f^{-1}(B_i)$ and G is open in X.

Further f(G) = $f\left(f^{-1}\left(\cap_{i=1}^{n} B_i\right)\right) = f\left(f^{-1}(B)\right) \subseteq B \subseteq Y$.

Hence f is continuous at X. Since this holds $\forall x \in X$,f is continuous on X.

8.2. WEAK TOPOLOGY :

8.2.1:Definition : Let P be a non-empty set. A partial order relation in p is a relation which is symbolized by $\leq$ and assumed to have the following properties.

Following Properties:

i) $x\leq x$ for every x in P (reflexive)

ii) $x\leq y$ and $y\leq x$ imply x=y (anti- symmetry)

iii) $x \leq y$ and $y \leq z$, $x \leq z$ (transitivity) if $\leq$ is a partial order in P then $(P,\leq)$ is called a partially ordered set.

8.2.2 : Definition : Partially ordered set $(L, \leq)$ is said to be a lattice. If for each pair of elements there exists a supremum and infimum in L. If a, b$\in$ L then the supremum of a and b is denoted by avb. The infimum of a and b is denoted by $a\wedge b$.

8.2.3 : Definition : Let $(L, \leq)$ be a lattice. If there exists an element denoted by I,L such that $x \leq I \forall x \in L$, then I is called a supremum of the all element of L. If there exists an element, denoted by O in L such that $0 \leq x \forall x \in L$ then the element O is called the zero element in L.

8.2.4 : Definition :A Lattice $(L, \leq)$ is called a complete lattice, if every infinite subset of L has a supremum and infimum in L.

8.2.5: Remarks :

1.If $(L,\leq)$ is a complete lattice then ''L'' contains the zero element and the all element.

2.A lattice $(L,\leq)$ is a complete lattice $\Leftrightarrow (L,\leq)$is a lattice with all element I and every non-empty subset of L has the infimum in L.

8.2.6 : Definition : Let X be a non-empty set and let T_1 and T_2 be two topologies on X. We say that T_1 weaker than T_2 (or) T_2 is stronger than T_1 if $T_1 \subseteq T_2$.

8.2.7 : Definition : Two topologies T_1 and T_2 on a non-empty set X are said to be comparable if either $T_1 \subseteq T_2$ (or)$T_2 \subseteq T_1$.

8.2.8 : Example : Let X= {a,b,c} and let $T_1 = \{\emptyset, X, \{a\}\}$

$T_2=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$

Then $T_1 \subseteq T_2 \Rightarrow T_1$ is weaker than T_2

Also,T_1 and T_2 are comparable

If $T_3 = \{\emptyset, X, \{b\}\}$ then $T_1 \nsubseteq T_3$ are not comparable

We have $T_3 \subseteq T_2$

So,T_3 is weaker than T_2

$\Rightarrow T_3$ and T_2 are comparable.

8.2.9 : Theorem : Let X be a non-empty set. Then the collection of all topologies on X forms a complete lattice under the relation "is weaker than".

Proof : Let X be a non-empty set and let τ be the collection of all topologies on X. Let $T_1=\{\emptyset, x\}$. Then T_1 is a topology on X and this topology is called the indiscrete topology on X.

$\Longrightarrow T_1 \in \tau$.

Let T_D be the class of all subjects of X.(i.e, the power set of x) T_D is a topology on X and this topology is called the discrete topology on X$\Longrightarrow T_D \in \tau$.

$\Rightarrow \tau$ is non-empty.

Now define a relations'' is weaker than'' which is symbolized by the symbol $\leq$ as follows:-

If $T_1,T_2 \in \tau$ and $T_1 \leq T_2 \Leftrightarrow T_1 \subseteq T_2$.

To prove that $\leq$ is a partial order :

i)Reflexivity :

Let $T \in \tau$, we have

$T \subseteq T \Longrightarrow T \leq T$

$\therefore \leq$ is reflexive on T

ii)Anti-Symmetry:

Let $T_1,T_2 \in \tau$ and suppose that

$T_1 \leq T_2$ and $T_2 \leq T_1$

$\Rightarrow T_1 \subseteq T_2$ and $T_2 \subseteq T_1$

$\Rightarrow T_1=T_2$

$\therefore \leq$ is anti-symmetric on τ

iii)Transitivity :

Let $T_1,T_2,T_3 \in \tau$ and suppose that

$T_1 \leq T_2$ and $T_2 \leq T_3$

$\Rightarrow T_1 \subseteq T_2$ and $T_2 \subseteq T_3$

$\Rightarrow T_1 \subseteq T_3$

$\Rightarrow T_1 \leq T_3$

$\therefore \leq$ is transitive on τ.

By all these properties , $\leq$ is a partial order on τ and hence(τ,≤) is a partially ordered set.

To prove that(T,≤) is a lattice :

Let $T_1,T_2 \in \tau$

$\Rightarrow T_1$ and T_2 are two topologies on X.

$\Rightarrow T_1 \cap T_2$ is a topology on X

$\Rightarrow T_1 \cap T_2 \in \tau$

We have $T_1 \cap T_2 \subseteq T_1$ and $T_1 \cap T_2 \subseteq T_2$

$\Rightarrow T_1 \cap T_2 \leq T_1$ and $T_1 \cap T_2 \leq T_2$

$\Rightarrow T_1 \cap T_2$ is lower bound of T_1 and T_2

Let $T_3 \in \boldsymbol{T}$ be a lower bound of T_1 and T_2

$\Rightarrow T_3 \leq T_1$ and $T_3 \leq T_2$

$\Rightarrow T_3 \subseteq T_1$ and $T_3 \subseteq T_2$

$\Rightarrow T_3 \subseteq T_1 \cap T_2$

$\Rightarrow T_3 \leq T_1 \cap T_2$

$\therefore T_1 \cap T_2$ is the greatest lower bound of T_1 and T_2

Thus each pair or elements in τ has a g.l.b in τ

Let $T_1, T_2 \in \tau$

Let A = $\{T \in \tau / T_1 \subset T \text{ and } T_2 \subset T\}$

Let $T^* = \bigcap_{T \in A} T \Rightarrow T^* \in \tau$

Since $T_1 \subset T$ and $T_2 \subset T$ $\forall T \in A$

We have Let $T_1 \subseteq \bigcap_{T \in A} T$ and $T_2 \subseteq \bigcap_{T \in A} T$

$\Rightarrow T_1 \subseteq T^*$ and $T_2 \subseteq T^*$

$\Rightarrow T_1 \leq T^*, T_2 \leq T^*$

$\Rightarrow$Let T^1 be an upper bound of T_1 and T_2 in τ

$\Rightarrow T_1 \leq T^1 \text{ and } T_2 \leq T^1$

$\Rightarrow T_1 \subseteq T^1 \text{ and } T_2 \subseteq T^1$

$\Rightarrow T^1 \in A$

$$\Rightarrow T^* = \bigcap_{T \in A} T \subseteq T$$

$\Rightarrow T^* \leq T^1$

$\therefore T^*$ is the least upper bound of T_1 and T_2

Thus each pair of element in τ has and least upper bound in τ.

$\therefore (\tau, \leq)$ is a lattice.

To prove that $(\tau, \leq)$ is a complete lattice :

We have $T \subseteq T_D$ $\forall T \in \tau$

$\Rightarrow T_1 \leq T_D \forall T \in \tau$

$\Rightarrow T_D$ is the all element in τ

We have $T_1 \subseteq T \forall T \in \tau$

$\Rightarrow T_1$ is the zero element in τ

Let B be a non empty sub collection of τ

Then $T^* = \bigcap_{T \in B} T$ is the g.l.b of B. Thus, every non-empty sub class of τ has g.l.b in τ

Hence $(\tau, \leq)$ is a complete lattice with the zero element T_1 and all the element T_D

This completes the proof of the theorem.

8.2.10 :Problem : If f and g are continuous real or complex functions defined on a topological space X, then f+g, af and fg are also continuous functions. Further more, if f and g are real then f$\wedge$ g and f$\vee$g are continuous.

Proof : We shall prove that fg is a continuous at an arbitrary point x0$\in$ X, Let $\in > 0$ be given choose ε_1 such that $\varepsilon_1(|(x_0)|) + (|(g(x_0))|) + \varepsilon_1^2 < \varepsilon$.

Since f is continuous at x_0 there exists a neighborhood G_1 of X_0 such that

$x \in G_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_1$

Similarly there exists a neighborhood G_2 of X_0 such that

$x \in G_2 \Rightarrow |g(x) - g(x_0)| < \in_1$ $\ if\ G = G_1 \cap G_2$ then G is a neighbourhood of x such that

$x \in G \Rightarrow |(fg)(x) - (fg)(x_0)| = |f(x)g(x) - f(x_0)g(x_0)|$

$\leq |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)|$

$= |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)|$

$\leq \in_1 |f(x) - f(x_0)| + \in_1 |f(x_0)| + \in_1 |f(x_0)|$

$< \varepsilon_1^2 + \varepsilon_1(|f(x_0)| + |g(x_0)|) < \varepsilon$

Therefore fg is continuous at every $x_0 \in$ X.

Next, we shall prove that f$\vee$ g is continuous. We first observe that the sets of the form A=(a,∞) and B=(-∞,b) form an open sub base for the real line and by showing that the inverse image of any set is open.

We have $(f \vee g)^{-1}(A) = \{x: max\{f(x), g(x)\} > a\}$

$= \{x: f(x) > a\} \cup \{x: g(x) > a\}$

This follows because the sets on RHS are $f^{-1}(a,\infty)$ are open sets of X (note that f and g are continuous functions and (a,∞) is open).

Therefore $(f \wedge g)^{-1}(A)$ is open. In a similar way we can prove that (f$\wedge$ g) is also continuous on X.

8.2.11: Lemma : Let X be a topological space and le {f $_n$ } be a sequence of real or complex functions defined on x which converges uniformly to a function f converges on x, if all f_n 's are continuous, then f is also continuous.

Sol : We shall show that f is continuous at an arbitrary point X_0 in X. Since $f_n \to f$ uniformly, given $\varepsilon > 0$there exists an integer n_0.Such that all $x \in X, \left|f(x) - f_{n_0}(x)\right| < \frac{\varepsilon}{3}$.

Since $f_{n_0}$is continuous and thus continuous at x_0, There exists a neighborhood G of x_0 such that for all $x \in G, \left|f_{n_0}(x) - f_{n_0}(x)\right| < \frac{\varepsilon}{3}$.

There fore

$x \in G \Rightarrow |f(x) - f(x_0)| \leq \left|f(x) - f_{n_0}(x)\right| + \left|f_{n_0}(x) - f_{n_0}(x)\right| + \left|f_{n_0}(x) - f(x_0)\right|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

Hence the theorem.

8.3 : EXERCISE :

8.3.1: Let X be a non-empty set and $\{X_i\}$ be a non-empty class of topological spaces. If for each i there is given mapping f_i of X into X_i denoted by T is the weak topology on X generated by f_i's.

a) Show that T equals the topology generated by the class of all inverse images in X of open sets in X_i 's.
b) If an open sub base is given in each X_i , show that T equals the topology generated by the class of all inverse images in X of sub basic open sets in the X_i 's.
c) If Y is a sub space of the topological space (X, τ) Show that the relative topology on Y is the weak topology generated by the restrictions of f_i 's to Y.

8.3.2 : In each of the following, we specify a set $\{f_i\}$ of real functions defined on the real line R. In each case given a complete description of the weak topology on R generated by the f_i's

a) $\{f_i\}$ consists of all constants functions.
b) $\{f_i\}$ consists of a single function f, defined by
$f(x) = 0$, if $x \leq 0$ and $f(x) = 1$, if $x > 0$.
c) $\{f_i\}$ consists of a single functions f defined by $f(x) = -1$, if $x<0, f(0)=0$ and $f(x) = 1$, if $x > 0$.
d) $\{f_i\}$ consists of a single function f, defined by $f(x) = x$ for all x.

e) $\{f_i\}$ consists of all bounded functions which are continuous with respect to the usual topology on R.
f) $\{f_i\}$ consists of all functions which are continuous with respect to the usual topology on R.

8.3.3: Let X,Y be topological spaces $Z \subseteq X$. If $f : X \to Y$ is continuous .
a) The restriction g of f defined by $g(x) = f(x)$ for $x \in z$ is continuous on Z
b) $f : X \to f(X)$ is continuous.

8.3.4 : Let X,Y,Z be topological spaces; $f : X \to Y$ and $g : Y \to Z$ be continuous show that g o f is continuous.
8.3.5 : Give an example of a continuous map which is not open.
8.3.6 : Give an example of an open map which is not continuous.
8.3.7 : If $f: X \to Y$ is a bijection show that f is open if and only if f^1 is continuous.
8.3.8 : Let $n \geq 2$: $f_i : \mathbb{R}^n \to \mathbb{R}$ be a mapping for $1 \leq i \leq n$.
Define $f(x) = (f1\ (x), f2\ (x), \ldots\ldots\ldots\ldots, f\,n\ (x))$ show that f is continuous if and only if for each I, $1 \leq i \leq n, f_i : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

8.3.9 : If X and Y topological spaces write $X \sim Y$ is there if a homeomorphism from X onto Y. Prove the following:

a) $X \sim X$
b) $X \sim Y \Rightarrow Y \sim X$
Because of this symmetry X and Y are said to be homeomorphic if $X \sim Y$
c) $X \sim Y$ and $Y \sim Z \Rightarrow X \sim Z$.

8.4 SUMMARY :

Weak topology is a mathematical concept that describes the weakest topology on a space that makes certain functions continuous. It's a key concept in functional analysis, and is often used to study linear operators and functional.

8.5 TECHNICAL TERMS :

- Weak Convergence: Convergence of a sequence of function in the weak topology.
- Weak Limit: The limit of a sequence of functions in the weak topology.
- Weak Continuous: A function that is continuous with respect to the weak topology.
- Indiscrete Topology, also known as the trivial topology, on a set X is the topology that consists of only two open sets.

8.6 SELF ASSESSMENT QUESTIONS:

1. If $\tau 1$ and $\tau 2$ are two topologies on non-empty set X, then is topological space.
a. $\tau 1 \cap \tau 2$
b. $\tau 1 \cup \tau 2$
c. $\tau 1 \setminus \tau 2$
Ans: a

2. Which of the following statements are true for a metric topology (X, d).
a. arbitrary intersection of open set is open
b. arbitrary union of closed set is closed
c. arbitrary union of open set is open
Ans: c

3. What is the closure of the set $S = 1 - 1$ r˙ : $n \in N$ in usual topology on R
a. (0, 1)
b. [0, 1]
c. None of these
Ans; c

4. An indiscrete topology has only elements
a. 1
b. 2
c. 3
Ans: b

5. Which of the following is true for discrete topology
a. complement of any set if open is open
b. every set is a open set
c. both (a) and (b)
Ans: c

6. Which of the following is true for discrete topology on X
a. the topology coincides with the power set P(X)
b. Weaker than indiscrete topology on X
c. neither of (a) and (b)
Ans: a

7. Which of the following is true for a discrete topology on X
a. Weaker than any topology on X
b. only convergent sequences in discrete space are sequences that are eventually constant
c. both (a) and (b
Ans: b

8.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON -9

N147 COMPACT SPACES

OBJECTIVES:

The objectives of this lesson are to.

- To understand the concepts of open subsets and open covers in a topological space.
- To understand the concept of compact sets.
- To understand the concept of closed subspace of a topological space.

STRUCTURE:

9.0: INTRODUCTION:

It is well known that closed and bounded sets of real nembers have important properties in analysis. For example continuous real-valued functions defined on closed and bounded sets of real numbers are bounded and uniformly continuous. Incontrast to this, the function defined on the open unit interval (0,1) by f(x)= $\frac{1}{x}$ is neither bounded nor uniformly continuous.

An abstractization of this important property possessed by closed and bound sets of real numbers gives rise to the concept of compactness for topological spaces.

9.1: COMPACTSPACES:

9.1.1:Definition: Let X be a topological space. A class $\{G_i\}_{i\in I}$ of open subsets of X is said to be an open cover of, X if X = $\bigcup_{i\in I} G_i$. A sub class of an open cover which is itself an open cover is called a subcover. A topological space X is called a compact space if every open cover of X has a finite subcover. A subspaceYof a topological space X is said to be compact if Y is compact as a topological space in its own right.

9.1.2:Example:

1. Every indiscrete space is compact (Ex:5.1.4).

Solution: If X is an indiscrete space , since X has only two open sets, every open cover has a finite subcover. Thus X is compact.

2. Let X be any infinite set and let $T = \{V \subseteq X / X - V \text{ is finite }\} \cup \{\phi\}$ Then T is a topology on X, called cofinite topology; The space (X,T) is called a cofinite topological space.This cofinite topological space is compact (**Ex :** 5.1.6).

Solution: Let $\{V_i\}_{i\in I}$ be an open cover of X. Since X = $\bigcup_{i=1} V_i$. Some V_i is none empty say

V_{i_0}. Then X $-V_{i_0}$ is finite. Let X $-V_{i_0} = \{X_1 \dots X_n\}$. Suppose $X_r \in V_{i_0}$ is for $r = 1 \dots n$. Thus X = $V_{i_1} \cup V_{i_2} \cup \dots \dots \cup V_{i_n} \cup V_{i_0}$. So$\{V_{i_1}, \dots V_{i_n}, \dots V_{i_n}\}$ is a finite subcover. Hence X is compact.

3. Every finite topological space X (i.e|X|<∞) is compact.

Solution: Since X has only a finite number of open sets, every open cover has a finite sub cover.

∴ X is not compact.

4. The open unit interval (0,1) with usual topology is not compact.

Solution: For each positive integer n, Let $V_n = \left(\frac{1}{n}, 1\right)$. Then $\{V_n\}_{n\in\mathbb{N}}$ is an open cover of (0,1), but it has no finite subcover (0,1) is compact.

5. The set R of all real numbers with usual topology is not compact.

Solution: Clearly $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$. For each positive integer n, let $U_n = (-n, n)$.
Then $\{U_i\}_{i\in\mathbb{N}}$ is an open cover of $\mathbb{R}$, but has no finite subcover. Therefore $\mathbb{R}$ is not compact.

9.1.3: SAQ : Let Y be a subspace of a topological space X. Then Y is com- pact if and only if for every class $\{H_i\}_{i\in I}$ of open sets in X such that $Y \subseteq \bigcup_{i\in I} H_i$ there is a finite subclass $\{H_i\}_{i\in j}$ $(J \subseteq I \text{ and } J \text{ is finite})$ such that $Y \subseteq \bigcup_{i\in J} H_i$

We now prove two simple, but useful, theorems.

9.1.4:Theorem: Any closed subspace of a compact space is compact.

Proof: Let Y be a closed subspace of a compact space X . Let $\{H_i\}_{i\in I}$ be a Class of open sets in X such that $Y \subseteq \bigcup_{i\in I} H_i$.

Then $X = Y \cup Y^1 \subseteq (\bigcup_{i\in I} H_i) \cup Y^I \subseteq X$, where Y^1 is the complement of Y in X.

∴ $X = (\bigcup_{i\in I} H_i) \cup Y^1$. Since Y is closed, Y^1 is open.

Hence the class $\{H_i\}_{i\in I} \cup \{Y^1\}$ is an open cover of X. Since X is compact,

There exists a finite subclass $\{H_{i_1}, \dots\dots\dots, H_{i_n}\}$ of $\{H_i\}_{i\in I}$ such that $X = H_{i_1} \cup \dots \cup H_{i_n} \cup Y^1$.

Hence $Y = (H_{i_1} \cap Y) \cup \dots \cup (H_{i_n} \cap Y) \cup (Y^1 \cap Y) \subseteq H_{i_1} \cup \dots \cup H_{i_n}$.

By SAQ 9.1.3, Y is compact.

9.1.5 :Theorem: Any continous image of a compact space is compact.

Proof : Let $f: X \to Y$ be a continous mapping of a compact space X into an arbitrary topological space Y. We claim that f(X) is a compact subspace of Y. Let $\{H_i\}_{i\in I}$ be a class of open sets in Ysuch that $f(X) \subseteq \bigcup_{i\in I} H_i$. Since f is continous and Hi is open in Y. $f^{-1}(H_i)$ is open in X, for every $i \in I$ .Therefore $\{f^{-1}(H_i)\}_{i\in I}$ is a class of open sets in X.

Also $f(X) \subseteq \bigcup_{i\in I} H_i \Rightarrow X \subseteq f^{-1}(\bigcup_{i\in I} H_i) = \bigcup_{i\in I} f^{-1}(H_i) \Rightarrow X = \bigcup_{i\in I} f^{-1}(H_i)$. since X is compact, there exists a finite subcover $\{f^{-1}(H_{i_1}), \dots, f^{-1}(H_{i_n})\}$ of $\{f^{-1}(H_{i_n})\}$. Hence $X = f^{-1}(H_{i_1}) \cup \dots \cup f^{-1}(H_{i_n})$ and this implies that $f(X) \subseteq H_{i_1} \cup \dots \cup H_{i_n}$. Thus f(X) is compact.

9.1.6 : Remark: Let us recall if X is a set and $\{A_i\}_{i\in I}$ is a class of subsets of X, then we have that $(\bigcup_{i\in I} A_i)^1 = \bigcap_{i\in I} A_i^1$ and, $(\bigcap_{i\in I} A_i)^1 = \bigcup_{i\in I} A_i^1$

We note the following :

$\{A_i\}_{i\in I}$ is a covering of X $\Leftrightarrow X = \bigcup_{i\in I} A_i$
$\Leftrightarrow X - \bigcup_{i\in I} A_i = \phi$
$\Leftrightarrow (\bigcup_{i\in I} A)^I = \phi$
$\Leftrightarrow \bigcap_{i\in I} A_i^I = \phi$
We also note that a subset A of a topological space X is open iff its compliment A^I is closed. The following theorem is an easy consequence of the definition of compactness of a topological space.

9.1.7 :Theorem : A topological space is compact$\Leftrightarrow$every class of closed sets with empty intersection has a finite subclass with empty intersection.

9.1.8:Remark: In remark 9.1.6, it was observed that if {Ai} is a class of Subsets of a set X then $\{A_i\}_{i\in I}$ is a covering of X if and only if $\bigcap_{i\in I} A_i^I = \phi$. As a consequence of this we have that the class {Ai} is not a covering of X if and only if $\bigcap_{i\in I} A_i^I \neq \phi$

9.1.9 :Definition : A class $\{A_i\}_{i\in I}$ of subsets of non-empty set X is said to have the finite intersection property (simplify f.i.P), if every finite subclass of $\{A_i\}_{i\in I}$ has non empty intersection.

9.1.10:Theorem : A topological space is compact$\Leftrightarrow$every closed sets with the finite intersection property has non-empty intersection. Let us recall that an open base for a topological space X is a class of open sets with the property that every open set is a union of sets in this class .

9.1.11: Definition : Let X be a topological space .An open cover of X whose sets are all in some given open base is called a basic open cover .

9.1.12: Remark: Sppose X is a compact space. Since every basic open set is an open set, every basic open cover is an open cover and hence it has a finite subcover. We prove the converse part in the following.

9.1.13: Theorem: Suppose $\{B_i\}_{i\in I}$ is an open base for a topological space X. If every basic epen cover by sets from $\{B_i\}_{i\in I}$ has finite subcover, then X is compact.

Proof : Let $\{G_i\}_{i\in j}$ be an open cover of X. Since $\{B_i\}_{i\in I}$. Is an open base, each G_i is a union of sets from $\{B_i\}_{i\in I}$. So there is subset $I_r \subseteq I$ such that $G_r = \bigcup_{i\in I} B_i$. Put $I_0 = \bigcup_{i\in j} I_r$. Therefore $X = \bigcup_{i\in j} G_r = \bigcup_{i\in j}(\bigcup_{i\in I_r} B_i) = \bigcup_{i\in I_0} B_i$. Thus $\{B_i\}_{i\in I}$ is a basic open cover of X. By hypothesis, there exists a finite subclass $\{B_{i_1} \dots\dots B_{i_n}\}$ of $\{B_i\}_{i\in I_0}$such that $X = \bigcup_{k=1}^{n} B_{i_k}$. Now for each B_{i_k} , there exists $G_{i_k}(r_k \in J)$ such that $B_{i_k} \subseteq G_{r_k}$. So $X = B_{i_r} \cup B_{i_n} \subseteq G_{r_1} \cup \dots\dots \cup G_{r_n}$ and hence $X = G_{r_1} \cup \dots \cup G_{r_n}$. Thus X is compact.

9.1.14:Definition: Let X be a topological space. A class $\{F_1\}$ of closed subsets of X is called a closed base if the class $\{F_i^I\}$ of all complements of its sets is an open base of X. Sets F_i are called basic closed sets.

Theorem 9.1.13can be restated as follows.

9.1.15:Theorem: A topological space is compact if every class of basic closed sets with the finite intersection property has non-empty intersection.

9.1.16:Definition: Let X be topological space. A class $\{S_i\}$of closed subsets of X is called a closed subbase, if the class $\{S_i^1\}$ of all complements of its sets is an open subbase.

9.1.17:Remark: Let us recall that an open subbase is a classes of open subsets of a topological space X whose finite intersections from an open base. This open base is called the open base generated by the open subbases. From the definitions 9.1.14 and 9.1.16, it is clear that the class of all finite unions of sets in a closed subbase C is a closed base. This is called the closed base generated by the closed subbase C.

We now prove a criterion for a topological space to be compact in terms of subbasic closed sets.

9.1.8: Theorem: A topological space is compact if and only if every class of sub basic closed sets with finite intersection property has non-empty intersection.

Proof: Let X be a topological space. Since very subbasic closed set is a closed set, it follows from theorem 9.1.10 that if X is compact then every class of sub basic closed sets with f.i.p. has non-empty intersection. Conversely suppose that every class of subbasic closed sets with f.i.p has nonempty intersection, Let $\{S_\alpha\}_{\alpha\in\Delta}$ be a closed subbase and let $\{B_i\}_{i\in I}$ d be the closed base generated by this subbase. So, each B_i is a finite union of $S_\alpha^1 s$. By theorem 9.1.15, to prove the theorem it suffices to show that every class of basic closed sets from $\{B_i\}_{i\in I}$ with f.i.p. has non-empty intersection: So, let $\{\beta_i\}_{i\in J}$ be a class of B_i's with f.i.p. We have to show that $\bigcap_{i\in J} B_r \neq \phi$ Let $\Im_1$be the family of all classes of B_i's which contain $\{B_i\}_{i\in J}$ and have the f.i.p. Since the class $\{B_i\}_{i\in J}$ is in $\Im_1$, the family $\Im_1 \neq \phi$.Then $\Im_1$is a partially ordered set with respect to class inclusion. Let $\{B_i\}$ be a chain in $\Im_1$. Put $B = \bigcup_n B_N$ Since each B_N is a class of $B_i^I s$, is also a class of $B_i^I s$. Let $\{B_{i_1}, \ldots, B_{i_n}\}$ be a finite B_N class of sets in B, contained in some B_N. Since B_N has the f.i.p $B_{i_i} \cap \ldots \cap B_{i_n} \neq \phi$ Since $\{B_N\}$ is a chain. The finite class $\{B_{i_1}, \ldots, B_{i_n}\}$ so B has the f.i.p. Therefore $B \in \Im_1$ and is an upper bound of B_N. By Zorn's lamma, $\Im_1$ has a maximal element. Let $\{B_k\}_{k\in K}$ be a maximal clement in $\Im_1$ Since $\{B_k\}_{k\in K}$ contains $\{B_r\}_{r\in j}$, We have that $\bigcap_{k\in K} B_k \subseteq \bigcap_{i\in J} B_i$. So, it suffices if we show that $\bigcap_{k\in K} B_k \neq \phi$. Now consider the class $\{B_k\}_{k\in K}$. Each B_k is a finite union of sets in $\{S_\alpha\}_{\alpha\in\Delta}$ for instance let $B_k=S_1 \cup \ldots \cup S_n$. It now suffices to show that at least one of the sets $S_1 \ldots, S_n$ belongs to (*) the class $\{B_k\}_{k\in K}$. For, if we obtain such a set S_{α_k} for each Bk , then the resulting class $\{S_{\alpha_k}\}$is a class of subbasic closed sets. Since S_{α_k} is a $\bigcap_{k\in K} B_k \neq \phi$ Since $S_{\alpha_k} \subseteq B_k$ we prove (*) by contradiction. We assume that each of the sets $S_1 \ldots S_n$ is not in the class $\{B_k\}$. Consider S_1. Since each subbasic closed set is a basic closed set, S_1 is a basic closed set.

Since S_1 is not in the class $\{B_k\}$ the class $\{B_k\}_{k\in K} \cup \{S_1\}$ contains the class $\{B_k\}_{k\in K}$ poperly. By the maximality property of $\{B_k\}_{k\in K} \cup \{S_1\}$ fails to have the f.i.p. So there exists a finite subclass Γ_1 of $\{B_k\}_{k\in K}$ such that $S_1 \cap (\bigcap_{B\in\Gamma} B) = \phi$ if we do this process for each of the sets $S_1 \ldots S_n$ we get finite subclasses $\Gamma_1 \ldots \Gamma_n$ of $\{B_k\}_{k\in K}$ such that $S_i \cap (\bigcap_{B\in\Gamma} B) = \phi$. for $1 \le i \le n$ put $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$. Now Γ is a finits subclass of $\{B_k\}_{k\in K}$ such that $B_k \cap (\bigcap_{B\in\Gamma} B) = (S_1 \cap (\bigcap_{B\in\Gamma} B)) \cup \ldots \cup (S_{11} \cap (\bigcap_{B\in\Gamma} B)) = \phi$.

There fore $\Gamma \cup \{B_k\}$ is a finite subclass of $\{B_k\}_{k\in K}$ with empty intersection. This contradicts the finite intersection property of the class $\{B_k\}_{k\in K}$. Therefore one of the sets $S_1 \ldots, S_n$ belongs to the class $\{B_k\}_{k\in K}$ as defined.

By remark 9.1.8. the above theorem can be restated as follows.

9.1.19: Theorem: A topological space is compact if every subbasic open cover has a finite sub cover.

9.1.20: SAQ: Let X be a topological space and Y be a subspace of X If $\{V_i\}_{i\in I}$ is an open subbase for X then the class $\{U_i\}_{i\in I}$,where $U_i = V_i \cap Y$, $\forall i \in I$,is an open subbase of Y. We now prove the famous Heine-Borel theorem,

9.1.21:Theorem: (The Heine - Borel theorem) Every closed and bounded subspace of the real line is compact.

Proof: Let E be a closed and bounded subspace of the real line R. E is bounded $\Rightarrow E\subseteq[-n,n]$ some positive integer n. Since E is closed in R, it. Is also closed in [-n,n]. By theorem 9.1.4, to show that E is compact, it suffices to show that every interval of the form [a, b] is compact. If a = b, then [a, b] = {a} and hence it is compact, because every finite space is compact. So, we may assume that a<b. Clearly the class of all intervals of the form (c,+∞) and (-∞,d). where c and d are real numbers is an open base for R. By SAQ 9.1.20 by droping the empty set, the class or all intervals of the form [a,d) and [c,b] where c and d are real numbers such that a < c < b and a < d < b is an open subbase for [a.b], Therefore the class of all interals of the form [a,c] and [d,b], where a < c, d < b is a closed subbase for [a,b]. Let $y = \{[a, c_i]\}_{i\in I} \cup \{[d_i, h]\}_{i\in j}$ be a class of sub basic closed sets with f.i.p. It suffices to show that the intersection of all sets in Y is non empty.

If Y contains only intervals of the form [a,ci] then the intersection contains a. Similarly Y contains only intervals of the form [di, b], then the intersection contains b. So, we may assume that Y contains only intervals of the form both the types. Define Define $d = \sup\{d_i, [d_., b] \in Y\}$.Clearly $d \in [d_i, b]\ \forall i$. We complet the proof by showing that $d \leq c_i \forall i$. Suppose that $d > c_{i_0}$ for some i0. Then c_{i_0}, is not an upper bound of the defining set of d.

∴ There exists a d_{r_0} such that $c_{i_0} < d_{r_0}$. Thus $[a, c_{i_0}] \cap [d_{r_0}, b] = \phi$. This contradicts the f.i.p of Y. This completes the proof.

9.1.22: SAQ: Prove the converse of the Heine - Borel. theorem: Every compact subspace the real line is closed and bounded.

9.1.23: Definition: A topological space is said to be countably compact, if every countable open cover has a finite subcover.

9.1.24: SAQ: Prove that a second countable space is countably compact ⇔ it is cumpact.

9.2: MODEL EXAMINATTION QUESTIONS;

1. Prove that any closed subspace of a comact space is compact.
2. Prove that any continuous image of a compact space is compact.
3. Prove that a topological space is compact if and only it every class of basic closed sets with the f.i.p has non-empty intersection.
4. Prove that a topological space is compact if and only if every sub basic open cover has a finite subcover.

5. State and prove the Heine - Borel theorem.
6. Prove that every compact subspace of the real line is closed and bounded.

9.3: EXCERCISE:

1. Prove that a compact subspace of a metric spuce is closed and bounded.
2. Let X be a topological space. If Y_1 and Y_2 are compact subspaccs of X, prove That $Y_1 \cup Y_2$ is also a compact subspace of X.
3. If $\{X_i\}$ is a non-empty class of compact subspaccs of X each of which is closed and if $\cap X_i$ is non-empty. Show that $\bigcap_{i\in J} X_i$ is also a compact subspaces of X.
4. Show that a continous real or complex function defined on a compact space is bounded.
5. Show that a continuous real function defined on a compact space X attains its infimum and its supremum.
6. If X is a compact space, and if $\{f_n\}$ is a monotone sequence of continous real functions defined on X which converges pointwise to a continuous real funtion f defined on X. show that $\{f_n\}$ converges uniformly to f.
7. Prove9.1.6.
8. Prove that the class of intervals of the form (c,∞)or$(-\infty,d)$ where c, d are real numbers is an open base for R.

9.4 SUMMARY:

We learnt that the open subsets and open covers in a topological space. A closed subspace of a topological space. We have proved that closed subspace of a compact space is compact and closed and bounded subset of Euclidean space is compact.

9.5 TECHNICAL TERMS:

1. **Finite subcover;** A finite collection of open sets that cover a space.
2. **Open cover;** A collection of open sets that cover a space.
3. **Sub cover;** A subcollection of an open cover that still covers the space.

9.6 ANSWERS TO SELF ASSESSMENT QUESTIONS:

9.1.4: Suppose that Y is a compact subspace of X. Let $\{H_i\}_{i\in I}$ be a class of open sets in X such that $Y \subseteq \bigcup_{i\in I} H_i$.

Then $\{Y \cap H_i\}_{i\in I}$, is an open cover of Y. Hence, there exists a finite subcover, say $\{Y \cap H_{i_1}, \ldots\ldots y \cap H_{i_n}\}$. Therefore

$(Y \cap H_{i_1}) \cup \ldots\ldots \cup (Y \cap H_{i_n}) = Y \cap (H_{i_1} \cup \ldots . \cup H_{i_n}) \subseteq H_{i_1} \cup \ldots . \cup H_{i_n}$.

Since every open set G in Y can be written as $G = Y \cap H$, where H is open in X, the converse part can be proved in a similar way.

9.1.11: Let H be any non-empty open set in Y and $y \in H$. Then $H = G \cap Y$, where G is open in X. Since $y \in G$, there exists $V_{i_1}, \ldots, V_{i_n}$ in $\{V_i\}_{i\in I}$ such that $y \in V_{i_1} \cap \ldots . \cap V_{i_n} \subseteq G$.Then, clearly $y \in V_{i_1} \cap \ldots . \cap V_{i_n} \subseteq H$. $\therefore$ $\{U_i\}_{i\in I}$ is an open subbase for Y.

9.1.13: Let Y be a compact subspace of the real line R. For each positive integer n, In = (–n, n) Then, $\{I_n\}_{n\in N}$ is a class of open sets in R such that $Y \subseteq \bigcup_{n=1}^{\infty} I_n$ since Y is compact, there exists positive integers $n_i, \ldots\ldots. n_k$ such that $Y \subseteq I_{n_I} \cup \ldots \cup I_{n_r}$. Let n be the maximum of $n_i, \ldots. n_k$. Then $Y \subseteq I_n \Rightarrow Y \subseteq (-n, n) \Rightarrow Y$ is bounded. To show Y is closed it suffices to show that its complementary Y^1 is open. Let $x_0 \in Y^1$. For each $x \in Y$. Since $\neq x_0$, there exists neighbourhoods V_x of x and V_{x_0} of x_0 such that $V_x \cap V_{x_0}$ clearly $Y \subseteq \bigcup_{x\in Y} V_x$. Since Y is compact, there exists $x_I, \ldots\ldots, x_m \in y$ such that $Y \subseteq V_{x_i} \cup \ldots . \cup V_{x_m}$. Let $V_{x_i}, \ldots V_{x_m}$ be the corresponding neighbourhoods of x_0. Put $G = V_{x_i} \cup \ldots . \cup V_{x_m}$ and $H = V_{x_i} \cap \ldots . \cap V_{x_m}$. Then $x \in G \Rightarrow x \in V_{x_i}$ for some $i \Rightarrow x \notin V_{x_i} \Rightarrow x \notin H$. The $G \cap H = \phi$ and hence $x \in H \subseteq G^1 \subseteq Y^1$. Thus Y^1 is open.

9.1.24: Let X be a second countable space. Since every countable open cover is an open cover, it follows that if X is compact, then 'it is countably compact.

Conversely suppose that X is countably compact. Let $\{G_i\}_{i\in I}$ be an open cover of X.

Then $= \bigcup_{i\in I} G_i$. By Lindelof's Theorem there exists a countable subclass $\{G_{i_I}, G_{i_I}, \ldots\ldots\}$ such that $X = \bigcup_{i\in I} G_i$ thus $\{G_{i_r}\}_{r\in N}$ is countnble open cover of X. by hypothesis. There exists a finite subcover, say, $\{G_{i_I}, G_{i_I}, \ldots\ldots, G_{i_r}\}$. Since this is a finite subcover of $\{G_i\}_{i\in I}$ we have that X is compact.

9.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Dr. V. Amarendra Babu

LESSON-10
PRODUCT SPACES

OBJECTIVES:

The objectives of this lesson are to.

- To understand the concepts of the product topology on a product space.
- To understand the concepts of the subspace topological space.
- To understand the concepts of a open base for a topology.

STRUCTURE:

10.0: INTRODUCTION:

In this lesson we introduce the notions of product toplology and product spaces. We define these notions initially for two topological spaces for a better understanding and prove that the usual topology on the Euclidean plane R^2 is precisely the product topology. We then extend these notions to arbitrary class of topological spaces. We prove the main theorem of this lesson, namely the Trychonoff's theorem. As an application of this theorem we obtain the General- ized Heine-Borel Theorem. We also define the notion of locally compact space and some examples of these spaces are given.We obtain an equivalent condi- tion for a topological space to be locally compact.
We begin with the notions of product topology and product space for two topological Spaces.

10.1: PRODUCT SPACE :

Let X_1 and X_2 be topological spaces. Let us recall that the Cartesian product of the sets X_I and X_2 is the set of all ordered pairs (x_1,x_2) with $x_2 \in X_2$ and $x_1 \in X_I$ We denote it.by X_1x X_2Suppose X= X_1x X_2.Let S be the class of all subsets of X of the form G_1 x X_2 and X_1x G_2 where G_1 and G_2 are open subsets of X_1 and X_2 respectively. The topology on X. generated by the class S is called the product topology. The open sets in the product topology are the unions of finite intersections of sets in S. The set X equipped with the product topology is called the product space or the product of the spaces X_1 and X_2.The product topology has S as an open subbase.

lt is clear that $(G_1 x X_2) \cap (X_1 x G_2) = (G_1 \cap X_2)$ x $(X_2 \cap G_2) = G_1 x G_2$There fore the open base generated by S is the class of all subsets of the form G_1 x G_2. Where $G_1$and $G_2$are open in $X_1$and X_2 respectively.

Define mappings $P_i: X \rightarrow x_i$ by $P_i(x) = x_i$, i =1,2 .... for all $(x_1\backslash,x_2) \in X$,

P_1, P_2 are called the mappings of projection.
let us recall that if X is non-empty $T_1$and T_2 are topologies on X such that $T_1 \subseteq T_2$ we say that T_1, is weaker than T_2. Further, the family of all topologies on X is complete lattice with respect to the relation is weaker than.

10.1.1: Theorem: Let X_1 and X_2 be the topological spaces and let X be their product space. Then the projections p_i, for (i) = 1,2 are continuous. Moreover the product topology is the weakest topology for which the projections are continuous.

Proof: If G_1 is an open set in X_1 then $p^{-1}(G_1) = G_1 \times X_2$.Which is a subbasic open set X, so p_1 is continuous. Similarly p_2 is continuous. Suppose T_1 is a topology on X for which the projections p_1 and p_2 are continuous. Then for each pair of open sets G_1 and G_2 and X_1 and X_2 respectively, the set $G_1 \text{x } G_2 = (G_1 \times X_2) \cap (X_1 \times G_2) = p_1^{-1}(G_1) \cap p_2^{-1}(G_2)$ must be open in T, Since the projections are continuous with respect to T. Thus every set which is open in the product topology must be open in T.

10.1.2: Definition: A mapping ϕ from a topological space X into a topolclgical Space Y is called an open mapping, if $\phi(G)$ is open in Y whenever G is open in X.

10.1.3: SAQ: Prove that the projections $P_1$and $P_2$are open mappings. Let us recall that the Euclidean plane R^2 is a normed real linear space, where R^2 is the set of all ordered pairs (x_1, x_2) of real numbers, under coordinate wise operations and norm

given by $\|(x_1, x_2)\| = \sqrt{|x_1|^2 + |x_2|^2}$.

10.1.4: Theorem: The usual topology on the Euclidean plane R^2 is precisely the product topology of the usual topologies on R taken twice .

Proof: we know thut the, function d defined by
$d((r_1, s_1), (r_2, s_2)) = \sqrt{|r_1 - r_2|^2 + |s_1 - s_2|^2}$
is a metric on R^2. We have to show that the topology induced by the metric d is precisely the product topology. Suppose G is a subset of R^2 which is open with respect to the metric d, and let $(r, s) \in G$. Then there exists an $\epsilon > 0$ such that the open sphere $S\epsilon(r, s) \subseteq G$. Let $V = \frac{S\epsilon(r)}{\sqrt{2}}$ and $W = \frac{S\epsilon(s)}{\sqrt{2}}$ which are open sets in R containing r and s respectively. We assert that V x W $\subseteq$ G and this will show that G is open in the product topology. If (x,y) $\in$ V x W, then x $\in$ V and $y \in W$; that is $|r - x| < \frac{\epsilon}{\sqrt{2}}$ and $|s - y| < \frac{\epsilon}{\sqrt{2}}$.

Thus d(r,s),(x,y) $= \sqrt{|r - x|^2 + |s - y|^2} < \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon$ and so (x,y)$\in S_\epsilon$(r,s) $\subseteq$ G, as desired.

Now suppose G is a subset of R^2 which is open with respect to the product topology, and let (x.y) $\in$ G. Then there exists open sets V and W such that (x,y) $\in$ V x W $\subseteq$ G. Thus x $\in$ V and y $\in$ W, so there exist $\epsilon_x, \epsilon_y > 0$ such that $S_{\epsilon_x}(x) \subseteq V$ and $S_{\epsilon_y}(y) \subseteq W$. Let $\epsilon = \min\{\epsilon_x, \epsilon_y\}$, We claim that $S_\epsilon(x, y) \subseteq S_{\epsilon_x}\{x\} \text{ x } S_{\epsilon_y}(y)$ which will show that G is open with respect to the metric d, since $S_{\epsilon_x}\{x\} \text{ x } S_{\epsilon_y}(y) \subseteq V$ x W $\subseteq$ G Now if (r,s) $\in S_\epsilon(x,y)$ then $|x - r| \leq \sqrt{|x - r|^2 + |y - s|^2} < \epsilon \leq \epsilon_x$ and

$|y-r| \leq \sqrt{|x-r|^2+|y-s|^2} < \in \leq \in_y$, so (r,s) ∈ $S_{\in_r}$(x) x S. |y| as desired.
We now prove that the product of two compact spaces is compact.

10.1.5:Theorem; If X and Y are compact spaces, then their product space X x Y is also compact.

Proof: Let $\{W_\lambda\}_{\lambda \in \wedge}$ be an open covering of X x Y. We choose an x_0 in X and Consider $\{x_0\}$ x Y. Corresponding to each y in Y, there is a $\lambda(y) \in \wedge$ such that $(x_0, y) \in W_{\lambda(y)}$. Then there exists a basic open set Uy x V(y) such that $(x_0, y) \in U_y \times V(y) \subseteq W_{\lambda(y)}$
The class $\{V(y)\}_{y \in y}$ is an open covering of Y. Since Y is compact there exist y_1……,y_m∈Y such that $Y = V(y_1) \cup \ldots \cup V(y_m)$. Let$U_{y_1}$,…..,$U_{y_m}$. The corresponding neighbourhoods of x0. Put U(x0) $= U_{y_1} \cap \ldots\ldots \cap U_{y_m}$. Then we have U(x0) x V(yr) $\subseteq W_{\lambda(yr)}$. for r = 1……m and so U(x0) x Y $\subseteq W_{\lambda(y_1)} \cup \ldots\ldots \cup W_{\lambda(y_m)}$. It follows that corresponding to each x in X there is a neighbourhood U(x) of x and there are finitely many elements $\lambda(x,1) \ldots\ldots \lambda(x, m(x))$ in ∧ such that $U(x)$ x $Y \subseteq W_{\lambda(x,1)} \cup \ldots \cup W_{\lambda(x,m(x))}$.

Now the class $\{U(x)\}_{F_{j_i} \cap \ldots \cap F_{j_n} \neq \phi}$ is an open covering of X Since X is compact, it follows that there are elements x_1,…..,x_n, in X such that $X = U(x_1) \cup \ldots \cup U(x_n)$. So we have
X x Y $\subseteq (U(X_1)$x Y$) \cup \ldots \cup (U(X_n)$x Y$) \subseteq \cup_{r=1}^{n} \cup_{i=1}^{m(x_i)} W_{\lambda(x_i, i)}$
Thus $\left\{\{W_{\lambda(x_i,i)}\}_{i=1}^{m(x_i)}\right\}_{i=1}^{n}$ is a finite sub covering of X x Y. Therefore X x Y is compact.

10.1.6: SAQ: Prove that if X and Y are topological spaces such thut their product space X x Y is compact. then X and Y are compact.

We now extend the notion of product topology to arbitrary class of topological spaces. Let us recall that the Cartesian product $P_{i \in I} X_i$ of anon-empty class of sets $\{X_i\}_{i \in I}$ is the set of all mappings f of I into $\cup_{i \in I} X_i$ such that f(i)∈ X_i for every $i \in I$. If $f \in P_{i \in I} X_i$ then f is denoted by = $\{x_i\}_{i \in I}$, where f(i) = x_i for each i ∈ I. For each i ∈ I, the projection mapping p_i is the mapping from $P_{i \in I} X_i$ into X_i defined by $P_r(\{x_i\}_{i \in I}) = x_r$ for every $\{x_i\}_{i \in I} \in P_{i \in I} X_i$.

10.1.7: Definition:

(i) Let $\{x_i\}_{i \in I}$ be a non-empty class of topological spaces and let X =$P_{i \in I} X_i$ be the Cartesian product of the sets $\{x_i\}_{i \in I}$ For each i ∈ I, let P_i be the projection of X onto X_i. Let S be the class of all subsets of X of the form, S= $P_i^{-1}(G_i)$ where i ∈ I and G_i is an open subset of X_i. The topology on X generated by the class S is called the product topology. The set X together with the product topology on it is called a product space of the product of the spaces $\{X_i\}_{i \in I}$

(ii) A subset of X is open with respect to the product topology if and only if it is a union of finite intersections of sets in S. It is clear that S is an open subbase for the product topology and is called the defining open subbase.

(iii) A sub set B of X is in the defining open subbase $\Leftrightarrow B = P_i^{-1}(G_i)$, for some i ∈ I and some open subset G_i of X_i $\Leftrightarrow B = P_{i \in I}^{-1}(G_i)$, where, G_i = X_i for $i \neq i$ and G_i is an open set in X_i $\Leftrightarrow B = P_i^{-1}(G_i)$, where G_i is open subset of X_i which equals X_i for all i's but one. The class of all complements of open sets in the defining open subbase- namely, the class of all products of the fonn $P_{i\in I}F_i$, where F_i, is a closed subset of X_i which equals X_i for all i's but one - is called the define closed subbase .

(iv) The open base generated by the defining open subbase, that is, the class of all finite intersections of Sub basic open sets, is called the defining open base for the product topology. A subset G of X is in the defining open base if and only if it is of the form G $= P_{i\in I}G_i$,where G_i is an open subset of X_i which equals X_i for all but a finite number of i's.

As in theorem 10.1.1 one can prove that all the projection mappings P_i are continuous and the product topology is the weakest topology for which the projections are continuous. Also it is clear that all the projection mappings are open.

10.1.8: SAQ: Let f be a mapping of a topological space X into a product space $P_{i\in I}X_i$. Prove that f is continuous ⇔ p_i of is continuous for each projection Pi·

10.1.9:Definition: Let X be anon-empty set, let $\{X_i)$ be a non-empty. class of topological spaces, and for each i let f_i be a mapping of X into X_i. Note that if X is given its discrete topology, then all the f_i's are continuous. The intersection of all topologies on X with respect to each of which all the f_i's are continuous is called the weak topology generated by the f_i's.

It is clear that this is the topology on X which makes all the f_i^1 s continuous and it is the weak topology for which all the f_i^1 s are continuous.

10.1.10: Remark: in view of the above definition. il is obvious that if $\{X_i\}$ is a non-empty class of topological spaces and if $X = P_iX_i$is their product space, then. the product topology on X is the weak topology generated by the set of all projections .We now prove the main theorem of this lesson.

10.1.11:Theorem: (Tyconoff's theorem) Let $\{X_i\}_{i\in I}$ be a non-empty class of topologlical space and let $X=P_{i\in 1}X_i$ be their product space. Then X is a compact if and only if eache space X_i is compact.

Proof: It X is compact, then each space X_i compact space the projections are continuous and onto. Hence, suppose that each space X_i is compact. Let $\{F_i\}_{i\in I}$ be a non-empty class of closed sets from the defining closed subbase for te Product topology on X . Therefore each F_J is a product of the form $F_J=P_{i\in I}\ F_{ij,}$ where F_{ij} is a closed subset of X_i which equatls X_i for all i's but one. We assume that the class $\{F_j\}_{j\in J}$ has the finite intersection property. To show X is compact, It suffices to show that $\cap_{j\in J}F_{ij} \neq \phi$. For a fixed $i\in I$,we show that the class $\{F_{ij}\}_{j\in J}$, all are closed subsets of X_i, has the finite intersection property.

If $\{F_{ij_1}........F_{ij_n}\}$ is a finite subclass of $\{F_{ij}\}_{j\in J}$ then the corresponding subbasic closed sets $F_{j_1}......F_{j_n}$ form a finite subclass $\{F_j\}_{j\in J}$. Since $\{F_j\}_{j\in J}$ has the finite intersection property, we have that $F_{j_i} \cap \ldots\ldots \cap F_{j_n} \neq \phi$. Choose a point x in $F_{j_i} \cap \ldots\ldots \cap F_{j_n}$. suppose $x = \{x_i\}_{i \in I}$ where $x_i \in X_i$ for all i. For $1 \leq k \leq n, x \in F_{ij_k} = P_{i \in I} F_{ij_k} \Rightarrow x_i \in F_{ij_k}$. Therefore $x_i \in F_{ir_1} \cap \ldots \ldots \cap F_{ir_n}$ and so $F_{ir_1} \cap \ldots \ldots \cap F_{ir_n} \neq \phi$. Thus the class $\{F_{ij}\}_{j \in J}$ has the finite intersection propert. Since X_i is compact, $\bigcap_{j \in J} F_{ij} \neq \phi$. Choose point a_i in $\bigcap_{j \in J} F_{ij}$. Since $i \in I$ was arbitrary, we have that $a_i \in \bigcap_{j \in J} F_{ij}$ for all i. put $a = \{a_i\}_{i \in I}$. Thus $a_i \in F_{ij}$ for all I and for all j $\Rightarrow a \in P_{i \in I} F_{ij}$ for all j $\Rightarrow a \in F_j$ for all $j \Rightarrow \bigcap_{j \in J} F_j \Rightarrow \bigcap_{j \in J} F_j \neq \phi$, as desired.

10.1.12: SAQ : Show that the relative topology on a subspace of a product. space is the weak topology generated by the restrictions of the projections to that subspace.

Let us recall that the n-dimensional Euclidean space R^n is the normed real linear space, where R^n is the real linear space of all ordered n-tuples $x = (x_1\ldots\ldots x_n)$ of real numbers under co-ordinatewise operations and the norm is given $\|x\| = \sqrt{|x_1|^2 + \ldots\ldots + |x_n|^2}$. The topology on R^n obtained from the norm is called the usual topology. As in theorem10.1.4,one can prove that the product topology on R^n is precisely the usual topology.

We now prove an important consequence of Tychonoff s theorem namely the 'Generalized Heine-Borel theorem

10.1.13: Definition: Let Rn be the n- dimensional Euclidean space. If (a_i, b_i) is a bounded open interval or, the real line for each i=1..........n, then the subset of R^n defined by is called an open rectangle in R^n, Similarly if [ai, bi] is a closed interval on the real line for i=1….n then $P_{i=1}^{n}(a_i, b_i) = \{(x_1 \ldots . x_n)/a_i \leq x_i \leq b_i \text{ for each } i\}$ is called a closed rectangle in R^n.

10.1.14:Theorem: (The Generalized Heine-Borel theorem) Every closed and bounded subspace of R^n is compact.

Proof: Let E be a closed and bounded subspace of R^n, Since E is bounded, there exists a real number K > 0 such that $\|x\| \leq K$ for all x∈E. If $x = (x_1\ldots\ldots x_n) \in E$, then $|x_i| \leq \|x\| \leq K$ and hence $X_i \in [-K, K]$ for all i.

Thus $E \subseteq P_{i=1}^{n}[-r_i, r_i]$, where r_i = k for all i. Since E is closed in Rn, it is also closed in the subspace $P_{i=1}^{n}[-r_i, r_i]$

Thus E is a closed subspace of the closed rectangle $P_{i=1}^{n}[-r_i, r_i]$ .To show E is compact it suffices to show that each closed rectangle is compact as asubspace of R^n

Let $X = P_{i=1}^{n}[-a_i, b_i]$ be a closed rectangle in R^n, Each coordinate Space [a_i, b_i] is compact by the Heine-Borel theorem. Therefore, by Tyehonoff's theorem.

$X = P_{i=1}^{n}[-a_i, b_i]$ is compact with the product topology. So to show that X is compact as a subspace of Rn, if suffices to show that the product topology on X is the same as its relative topology as a subspace of Rn. By the above remarks, the product topology on Rn is the same as its usual topology. By SAQ 10.1.12 the relative topology on X is precisely the weak topology generated by the restrictions of the projections toX. It is clear that the restrictions of the projections on R^n to X are precisely the projections on X. Therefore the relative topology on X as a subspace of R^n is precisely the product topology on X. This is the desired result and the proof of the theorem is we now discuss about the desired complete. result and the proof of the theorem is complete.

10.1.15: Definition: A topological space is said to be locally compact if each of its points has a neighbourhood whose closure is compact.

10.1.16: Examples:

(i) Every compact space is locally compact. For, if X is a compact space and if $x \in X$, then X itself is a neighborhood of x such that $\bar{X}$ = X is compact. Thus X is locally compact.The following example shows that every locally compact space need not be compact.

(ii) Let R^n be the n-dimensional Euclidean space. If $x \in R^n$ and if S_r {x) is any open sphere centered on x then S_r (x) is a neighborhood of x. Since the closure $\overline{S_r(x)}$ is closed and bounded. by the Generalized Heine Borel theorem, $\overline{S_r(x)}$ is compact,Henee R^n is locally compact. But R^n is not compact.

(iii) Every discrete space is locally compact. Let us recall that a class of neighborhoods of a point is called an open base at the point if each neighborhood of the point contains a neighborhood in this class.

We now prove a necessary and sufficient condition for a topological space to he locally compact.

10.1.17:Theorem: *A* topological space is locally compact if and only if there is an open base at each point whose sets all have compact closures.

Proof: Let X be a topological space, Suppose that X is locally compact. Let x be a point in X, Let $\mathfrak{R}_x$ be the class of all neighborhoods of x whose closures are compact Since X is locally compact the class $\mathfrak{R}_x$. is non-empty,We prove that $\mathfrak{R}_x$ is an open base at x. Let G be any neighborhood of x. Since X . is locally compact. there is a neighborhood H of x such that its closure $\bar{H}$ is compact.

Clearly G∩H is a neighborhood of x and its closure $\overline{G \cap H}$ is compact, since $\overline{G \cap H}$ is a closed sub space the compact space $\bar{H}$ (theorem 9.1.4), Thus G∩H∈ $\mathfrak{R}_x$.
such that $x \in G \cap H \subseteq G$. Therefore $\mathfrak{R}_x$ is an open base $\mathfrak{R}_x$ at x whose set all have compact closures. Since X is a neighborhood B of x in $\mathfrak{R}_x$ such that $\in B \subseteq X$. Now B∈ $\mathfrak{R}_x$ implies $\bar{B}$ is compact. Thus X is locally compact.

Conversely suppose that there is an open base at each point whose sets all have compact closures. Let $x \in X$, then there exists on open base.

10.1.7: SAQ: Let P_x : X x Y→ X be the projection mapping. Then P_x is continuous and onto, Since continuous image of a compact space is compact. it follows that P_X (XxY) =X is compact. Similarly Y is compact.

10.1.8: SAQ: Suppose f is continuous. For each i. the projection mapping $P_{i \in I} X_i \to X_I$ is continuous. There fore P of is continuous. Conversely suppose defininig open subbase of the product topology on $P_{i \in I} X_i$. Then $= P_i^{-1}(G_i)$, for some i and some open set G_i in X_i.
Therefore $f^{-1}(S) = f^1\left(P_i^{-1}(G_i)\right) = (p_i o f)^{-1}(G_i)$ is open, since P_i of is continuous.
Thus f is continuous.

10.2: MODEL EXAMINATION QUESTIONS:

1. Prove thut the usual topology on the n dimensional Euclidean space R^n is the same as the product topology on it.
2. Slate and prove Tychonoff's theorem.
3. State and prove Generalized Heiene - Borel theorem.
4. (a) Define a locally compact space. Prove that every compact space is locally compact. Is the converse true. Justify your answer.
   (b) Prove that a topological space is locally compact if and only if there is an open base at each point whose sets all have compact closures.

10.3: EXERCISCS:

1. Let X and Y be topological space. If Y is compact, prove that projection mapping of X x Y onto X is a closed mapping (Let A and B be topological spaces. A mapping $f:A\rightarrow B$ is called a closed mapping if f(F) is closed in B wherever F is closed in A)
2. Prove that if X and Y are metric spaces with metrics d_1 and d_2 respectively then the mapping d defined by d ((a. b), (c,d)) = $\sqrt{d_1^2(a,c) + d_2^2(b,d)}$ is a metric on X x Y which induces the product topology.
3. Let X be a metric space with metric d. Prove that d is a continuous mapping of X x X into R.
4. Prove that a closed subspace of a locally compact space is locally compact.
5. (a) Let X, Y and Z be metric spaces and let f be a mapping of the product space X x Y into the space Z. Prove that f is continuous if and only if $x_n \rightarrow x$ and $y_n \rightarrow y$ implies f $(x_n,y_n) \rightarrow f(x, y)$
   (b) Show that if f is continuous, then for any y in Y. the mapping $f_y:X \rightarrow Z$ defined by f_y {x) = f(x.y) is continuous and for any x in X the mapping $x^f:Y\rightarrow Z$ defined by $x^f(y)$ = f(x,y) is also continuous. (If we regard f as a function f(x,y) of two variables x and y. it is customary to say that f is, jointly continuous in both the variables x and y whenever f is continuous from the product space X x Y into Z
   (c) What about the converse of the result stated in (b)? Justify your. answer.

10.4 SUMMARY:

We learrnt that the properties of open sets and closed sets in the Euclidean plane. We proved that product of two compact spaces is compact, the product of any collection of compact spaces is compact.

10.5 TECHNICAL TERMS:

1. **Open base;** A collection of open sets that generate the topology of space.
2. **Sub space;** A subset of a topological space equipped with the subspace topology.
3. **Topological space;** A set equipped with a topology, which defines the open sets of the space.

10.6 ANSWERS TO SELF ASSESSMENT QUESTIONS:

10.12: SAQ: Let $\{X_i\}$ be a non-empty class of topological Spaces and let X= P_i X_i be their product space. Suppose Y is a subspace of X. For each i. let P_i: X → X_i be the projection mapping and let P_i / Y :Y → X_i be the restriction of p_i to Y. The product topology on X is the topology generated by the class of all subsets of X of the form $P_i^{-1}(G_i)$. where i is an index element and G_i is an open set in X_i. Therefore the relative topology on Y is the topology generated by the class of all subsets of Y of the form $P_i^{-1}(G_i) \cap Y$. Where i is any index element and G_i is any open subset of X_i. It is clear that $\left[{P_i}/{Y}\right]^{-1}(G_i) = P_i^{-1}(G_i) \cap Y$

Henec the relative topology on Y is the weak topology generated by the restrictictions ${P_i}/{Y}$ of the projections P_1 to y.

10.13: SAQ: Let G be an open subset of X, If p_1 (G) = ϕ then clearly it is open Suppose P_1(G) $\neq \phi$. Let a ∈ P_1(G), then a = p_1(x,y), for some (x,y) $\in G$.Then there exists a basic open set $G_1 \times G_2$, where G_1 and G_2 are open in X_1 and X_2 respectively, such that (x,y) ∈ $G_1 \times G_2 \subseteq G$.Thus p_1(x,y) ∈ $p_1(G_1 \times G_2) \subseteq p_1(G)$,since $p_1(G_1 \times G_2)=G_1$.Hence p_1 is an open mapping. Similarly, p_2 is also an open mapping.

10.7 SUGGESTED READINGS:

Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON-11
COMPACT NESS FOR METRIC SPACES

OBJECTIVES:

The objectives of this lesson are to.

- To understand the concepts of a metric space and understand its properties.
- To understand the concepts of compactness in a metric space and understand its siginificance.
- Define and apply the lebesgue number

STRUCTURE:

11.0: Introduction
11.1: Compactness for metric spaces
11.2: Model examination questions
11.3: Exercise
11.4: Summary
11.5: Technical terms
11.6: Answer to self assessment questions
11.7: Suggested readings

11.0: INTRODUCTION:

The famous 'Bolzano - weierstrass theorem' (with Its converse) states that a non-empty subset E of the real line is compact if and only if every infinitesubset of E has a limit point in E. This motivates the concept of Bolzano -Weierstrass property for metric spaces. In this lesson, we define this concept for metric spaces and prove that a metric space is compact if and only if it has the Bolzano - weierstrass property. We also introduce the notion of sequentially compactness for metric spaces and prove that a metric space is compact if and only if it is sequentially compact. In this lesson, we further define the notion of totally boundedness for metric spaces and prove that a metric space is compact if and only if it is sequentially compact. In this lesson. we further define the notion of totally boundedness for metric spaces and prove that a metric space is compact if and only if it is totally bounded. In the sequel. we define the notion of Lebesgue number of an open cover in a metric space and prove that every open cover of a sequentially compact metric space has a Lebesgue number. By using this as a tool, we prove that any continuous image of a compact metric space is uniformly continuous.

First, let us define. the following very important concept.

11.1: COMPACTNESS FOR METRIC SPACES:

11.1.1: Definition: A metric space X is said to have the Bolzano - Weierstrass property if every infinite subset of X has a limit point in X.

11.1.2:Theorem: Every compact metric space has the Bolzano-Weierstrass property.

Proof: Assume that the metric space X is compact.We show That every infinite subset of X has a limit point in X. Suppose thatA is an infinite subset of X With no limit points. Since each point $x \in X$ is not a limit point in A, there exists an open sphere $S_{r_x}(x)$ centered on x such that $S_{r_x}(x) \cap A \subseteq \{X\}$.Since the class $\{S_{r_x}(x)\}$ – forms an open covering of X, there must be some finite Subcovering $X = \cup_{i=1}^{n} S_{r_x}(x_i)$. Therefore $A = A \cap X = \cup_{i=1}^{n}\left(A \cap S_{r_{x_i}}\right) \subseteq \{x_1 \ldots.. x_n\}$ and so A is finite. A contradiction thus every infinite subset of X must have a limit point in X.

11.1.3: SAQ: Prove that a compact subspace of a metric space is closed.

Let us recall the following definitions, If X is a metric space withmetric d and if x is a point and $\{x_n\}$ is a sequence in X, we say that the sequence $\{x_n\}$ has a limit or converges to x, written lim x_n= x or $x_n \to x$, if for every $\epsilon > 0$ there exists an integer N such that d $(x_n, x) < \epsilon$ whenever $n \geq N$. If $\{x_n\}$ is a sequence in X and if $\{n_k\}$ is a sequence of positive integers such that $n_1<n_2<$...........then the sequence $\{x_{nk}\}$ is called a subse quence of $\{x_n\}$.

11.1.4: Definition: A metric space is said to be sequentially compact if every sequence in X has a convergent subsequence.

11.1.5: Theorem: A metric space is sequentially compact if and only if it has the Bolzano weierstrass property.

Proof: Let X be a metric space. Assume that X is sequentially compact. We show that every infinite subset A of X has a limit point in X. Since A is infinite, we can choose a sequence $\{x_n\}$ of distinct points from A. Since X is sequentially compact. the sequence $\{x_n\}$ has a subsequence $\{x_{nk}\}$ which converges to a point x in X. Since $\{x_{nk}\}$ is a sequence of distinct points, x is a limit point of the set $\{x_{n_k}/k \geq 1\}$. Since the set $\{x_{n_k}/k \geq 1\} \subseteq$ A, it follows that x is a limit point of A.

Conversely suppose that every infinite subset of X has a limit point in X. We prove that X is sequentially compact. Let $\{x_n\}$ be an arbitrary sequence in X. If the sequence $\{x_n\}$ has a point x which is infinitelyy repeated, then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $x_{n_k} = x$ for all $k \geq 1$. This subsequence $\{x_{n_k}\}$, converges to x if no point of $\{x_n\}$ is infinitely repeated, the set A of points of the Sequence $\{x_n\}$ is infinite. Since A is infinite. It has a limit point x. Then each open sphere centered on X contains infinitely many points of A.We choose a subsequence $\{x_{nk}\}$ as follows. Choose n_1 such that d $(x, x_{nk}) < 1$. Having $n_1 \ldots n_{k-1}$ such that $n_1 < n_2 < \ldots. < n_{k-1}$ and d(x, $x_{n_d}) < \frac{1}{i}$ for $i = 1 \ldots. k-1$ choose an integer $n_k > n_{k-1}$ and d(x, $x_{n_k}) < \frac{1}{k}$ for all $k \geq 1$. Clearly $\{x_{n_k}\}$ converges to x. Thus X is sequentially compact.
Let us recall the following definition:

Let X be a metric space with metric d and let $A \subseteq X$, The diameter d(A) of A is defined by d(A) = sup{d(x,y) /x,y$\in$A}, A is said to have finite diameter if d(A) is a real number. In this case we Say that A is bounded, observe that $A = \phi$ if and only if d (A) $= -\infty$. So, if $A \neq \phi$, then O $\leq$ d (A) $\leq \infty$.

11.1.6: Definition: Let $\{G_i\}$ be an open cover of metric space X. A real number a >0 is called a Lebesgue number for the open cover $\{G_i\}$ if each subset of X whose diameter is less than a is contained in at least one G_i's

11.1.7:Theorem: (Lebesgue's covering lemma). In sequentially compact metric space, every open cover has a Lebesgue number.

Proof: Let X be a sequentially compact metric space and $\{G_i\}$ be an open cover of X. We say that subset of X is 'big' if it is not contained in any G_i. If there are no big sets, then any positive real number will serve as a Lebesgue number. We may thus assume that big sets do exist. Note that every big set contains at least two points. We define a^1 = glb (d(B)/B is a big set}. Clearly $0 \leq a^1 \leq \infty$ and $a^1 \leq d(B)$, for any big set B. It will suffice to show that $a^1 > 0$; for if $a^1 > 0$ then any real number a such that $0 < a < a^1$ will be a Lebesgue number. We therefore assume that $a^1 = 0$ and we deduce a contradiction from this assumption. For each positive integer n there exists a big set B_n such that $0 < d(B_n) < \frac{1}{n}$. Choose a point x_n in each B_n. Since X is sequentially compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, which converges to some point x in X. Then $x \in G_{i_n}$ for some set G_{i_n} in our open cover $\{G_i\}$. Since G_{i_n}, is open, there exists an open sphere $S_r(x)$ such that $S_r(x) \subseteq G_{i_n}$. Since $x_{n_k} \to x$, it follows that $x_{n_k} \in S_{r/2}(x)$ for infinitely many k, that is $x_{n_0} \in S_{r/2}(x)$ for infinitely many n.

Choose n_0 such that $x_{n_0} \in S_{r/2}$ and $n_0 > \frac{2}{r}$. Thus $0 < d(B_{n_0}) < \frac{1}{n_0} < \frac{r}{2}$. If $y \in B_{n_0}$, then $d(x,y) \leq d(x, x_{n_0}) + d(x_{n_k}, y)\frac{r}{2} + d(B_{n_0}) < \frac{r}{2} + \frac{r}{2} = r$.
Hence $B_{n_0} \subseteq S_r(x) \subseteq G_{i_0}$
This contradicts the fact that B_{n_0} is a big set.

11.1.8: Definition:

(i) Let X be a metric space and $\epsilon > 0$.A subset A of X is called an ϵ –net for X if A is finite and $X = \bigcup_{a\in A} S_\epsilon(a)$
(ii) A metric space X said to be totally bounced if it has an ϵ net for each $\epsilon > 0$.

Let us recall that a subset A of X is said to be bounded if $0 \leq d(A) < \infty$,where d(A) is the diameter of A.

11.1.9: SAQ: Prove that a totally bounded metric space is bounded.

11.1.10:Theorem: Every sequentially compact metric space is totally bounded.

Proof: Let X be a sequentially compact metric space with metric d. Suppose that X is not totally bounded. Then. for some $\epsilon>0$. X must have no ϵ- net. Let $x_1 \in X$.Then the finite set $\{x_1\}$ is not an ϵ- net for X, and so there exists a point $x_2 \notin S_\epsilon(x_1)$: Therefore $d(x_1,x_2) \geq \epsilon$.

Now the finite set $\{x_1,x_2\}$ is also not an ϵ-net, and so there exists a point $x_3 \notin \bigcup_{i=1}^{2} S_3(x_i)$. Thus $d(x_1, x_3) \geq \epsilon$ and $d(x_1, x_2) \geq \epsilon$. Proceed by induction. If there exist a set of points $\{x_1,\ldots,x_n\}$ such that $d(x_1,x_r) \geq \epsilon$ whenever $i \neq r$, then this finite set is not an ϵ–net and so there exists a point $x_{n+1} \notin \bigcup_{i=1}^{n} S_\epsilon(x_i)$; that is $d(x_1,x_r) \geq \epsilon$ whenever $i \neq r$. Now by induction, we have a sequence $\{x_n\}$ of distinct points in X such that $d(x_i,x_r) \geq \epsilon$ whenever $i \neq r$.

Since X is sequentially compact, the sequence $\{x_n\}$ has a subsequence (x_{n_k}). which converges to a point x∈X. But the open sphere $S_{\epsilon/2}(x)$ *must* contain x_{n_k} for k > N, where N is some positive integer; that is $S_{\epsilon/2}(x)$ contains x_n for infinitely many n. This contradicts the fact that $d(x_i, x_r) \geq \epsilon$ whenever i≠r. Hence X is totally bounded.

11.1.12: Theorem: Every sequentially compact metric space is compact.

Proof : Let X be a sequentially compact metric space. Let $\{G_i\}$ be an open cover of X. By lebessgue's covering lemma, the open cover $\{G_i\}$ has a Lebesgue number a. Put $\epsilon = a/3$. theorem11.1.10,X has an ∈–net, say $\{a_1,\ldots,a_n\}$. For each k = 1,2,....,n, we have that the diameter d $(S_\epsilon(a_k)) \leq 2\epsilon < a$ .Since a is a Lebesgue number of the open cover $\{G_i\}$, each k, there exists an open set G_{i_k} in $\{G_i\}$ such that $S_\epsilon(a_k) \subseteq G_{i_k}$.

Thus $X = S_\epsilon(a_i) \cup \ldots\ldots \cup S_\epsilon(a_n) \subseteq G_{i_1} \cup G_{i_2} \cup \ldots\ldots \cup G_{i_n}$ and hence $X = G_{i_1} \cup \ldots\ldots \cup G_{i_n}$. Therefore X is compact.

11.1.13:Theorem: If X is a metric space then the following conditions are equivalent,
(i) X is compact (ii) X is sequentially compact
(iii) X has the Bolzano - Weierstrass property

Proof : (i) ⇔ (iii) follows from the theorem11.1.2, (iii) ⇔ (ii) follows from the theorem 11.1.5 and (ii) ⇔ (i) follows from the theorem 11.1.11.

11.1.14: SAQ: Show that a compact metric space is separable.

Proof: We now prove an important theorem regarding continuous functions of compact metric spaces into arbitrary metric spaces.

11.1.15:Theorem: Any continuous mapping of a compact metric space into a metric space is uniformly continuous.

Proof: Let f be a continuous mapping of a compact metric space X into a metric Space Y. Let d_x and d_y be the metrics on X and Y respectively, Let ∈> 0. For each x ∈X, consider the open sphere $S_{\epsilon/2}$ (f(x)) centered on f(x) and radius $\frac{\epsilon}{2}$ in *Y,* Since f is continuous.$f^{-1}\left(S_{\epsilon/2}(f(x))\right)$ is an open set in X containing x, Now, the class $\left\{f^{-1}\left(S_{\epsilon/2}(f(x))\right)\right\}_{x \in X}$ is an open cover of X. since X is compact, this open cover has a Lebesgue number $\delta > 0$. If x, $x^1 \in X$ are such that $d_x(x, x^1) < \delta$, then the set $\{x, x^1\}$ is a set with diameter $< \delta$. Therefore $\{x, x^1\} \subseteq f^{-1}$. $\left(S_{\epsilon/2}(f(x_0))\right)$ for some $x_0 \in X$. Hence f(x), $f(x^1) \in S_{\epsilon/2}(f^1(x_0))$. This implies that d_x $(f(x).f(x^1)) \leq d_Y((x).f(x_0)) + d_Y(f(x_0).f^1(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ thus f is uniformly continuous.

11.2: MODCL EXAMINATION QUESTIONS:

1. Prove that a metric space is sequentially compact iff it has the Bolzano Weierstrass property.
2. Prove that every open cover of a sequentially compact metric space has a Lebesgue number.
3. Prove that evry compact topological space has the Bolzano –Weierstrass property.
4. Prove that every sequentially compact metric space is totally bounded.
5. Prove that every sequentially compact metric space is compact.

6. Prove that any continuous mapping of a compact of a compact metric, space is uniformly continuous.

11.3: EXERCISE:

1. Prove thate very compact topological space has the Bolzano-Weierstrass property.
2. Let X be a metric space with metric d and let $\epsilon > 0$. Prove that if x $\in$ X, then the set $M_\epsilon(x) = \left\{ y \in x \big/ d(x, y) > \epsilon \right\}$ is open in X.
3. Let X be the set of all positive integers. Let T be the topology on X generated by the class of all sets of the form {2n–1, 2n} , where n $\in$ X. Show that with this topology T, X has the Bolzano-Weierstrass property but it is not compact.
4. Prove that if E, is a compact subset of a metric space, then its derived set $\bar{d}(E)$ is also compact ($\bar{d}(E)$ is the set of all limit points of E in X).
5. Prove that a subspace A of a metric space X is totally bounded iff $\bar{A}$ is totally bounded.

11.4 SUMMARY:

We learrnt that the understand the Bolzano – Weierstrass property and its relation to compactness. We proved that the equivalence of sequential compactness and compact understand the relationship between completeness, totally boundedness and compactness.

11.5 TECHNICAL TERMS:

1. **Metric space;** A set equipped with a metric.
2. **Compact space;** A metric space where every open cover has a finite sub cover.
3. **Lebesgue number;** The largest number such that every subset of a metric space with diameter less than this number is contained in some member of an open cover.
4. **Open cover;** A collection of open sets that cover a metric space.

11.6 ANSWER TO SELF ASSESSMENT QUESTIONS:

11.1.3:SAQ: Let Y be a compact subspace or a metric x and let d be the metric on X. To prove that Y is closed, it suffices to show that its complement y^1 in X is open. Let Z ϵ y1. For each positive integer n, let $A_n = \left\{ x \in X / d(x,z) < \frac{1}{n} \right\}$.
Then, $\{A_n\}$ is an ascending sequence of open sets in X such that $Y \subseteq \bigcup_{n=1}^{\infty} A_n$.
Since Y is compact and $\{A_n\}$ is an ascending sequence, there exists a positive integer n such that $Y \subseteq A_n$ clearly $y \in S_{1/n}(z) \subseteq Y^1$ is open. Sphere with centre, z and radius $\frac{1}{n}$ Hence. Y^1 is open.

11.1.9:SAQ: d Let X be a totally bounded - metric space with metric d. Let $\epsilon > 0$. Since X is totally bounded, X has an ϵ– net, say {a1, a2…..,an}.
Then $X = \bigcup_{i=1}^{n} S_\epsilon(a_i)$.
If x, y $\in$ X then x $\in$ S_ϵ (a_i) any y $\in$ S_ϵ (a_i) for some i, r. Therefore $d(x,y) \leq d(x, a_i) + d(a_i, a_r) + d(a_r, y) \leq d(\{a_1 \ldots a_n\}) + 2\epsilon$ where $d(\{a_1 \ldots a_n\})$ is the diameter of

$\{a_1.....,a_n\}$. This implies that d (X) $\leq$ d (($a_1......a_n$})) + 2∈ < ∞.
Hence X is bounded.

11.1.13:SAQ: Let X be a compact metric space. By theorems 11.1.2 and 11.1.10 X is totally bounded. For each positive integer n, let C_n be an $\frac{1}{n}$–net of X. Put D $= \bigcup_{n=1}^{\infty} C_n$ Since each C_n is finite, it follows that D is countable.

To prove the result, it suffices to prove that D is dense in X. Let $S_r(x)$ be any open sphere in X. Choose n such that, $\frac{1}{n} < r$. Since C_n is an $\frac{1}{n}$ - net for X, we get that $X = \bigcup_{a \in C_n} S_{1/n}(a)$.

Therefore $x \in S_{1/n}(a)$ for some $a \in C_n \subseteq D$. since $\frac{1}{n} < r$, it follows that $a \in S_{1/n}(x) \subseteq S_r(x) \neq \phi$. Hence D is dense in X.

11.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Dr. V. Amarendra Babu

LESSON -12

ASCOLI'S THEOREM

OBJECTIVES:

The objectives of this lesson are to.

- To understand the concepts of a metric space.
- To understand the concepts of continuity in a metric space.
- To understand the concepts of compactness in a metric space.

STRUCTURE:

12.0: INTRODUCTION:

In Lesson 11 we established that compactness of a metric space is equivalent to sequential compactness as well as BolzanoWeierstrass property. The full power of these criteria becomes evident when these are found to be instrument - tal to characterize cmpact subsets of the space C(X,$\mathbb{C}$) of complex valued continuous functions on a compactmetric space X.This characterization is known as Ascoli's theorem also called Arzela - Ascoli theorem and Ascoli – Arzola theorem .

This theorem is based on "Cantor's diagonalization process" which en- ables us to select a sequence from an array. of sequences in such a way that except for a few terms in the beginning depending on the array all the remaining terms lie in every array.

12.0.1:SAQ: With notation as above for every m $\geq$ 1the sequence $\{y_m, y_{m+1} \ldots y_{m+n}, \ldots\}$ where $y_k = x_{k_{\epsilon_{k+n}}}$ $\forall k \geq 1$is a subsequence of S_m. In particular $\{y_1, y_2 \ldots y_k\}$ is a subsequence of $\{x_1, x_2 \ldots x_n \ldots\}$.

In the sequel (X,d) stands for a compact metric space and C (X, $\mathbb{C}$) for the Branch space of all complex valued continuous functions on X.

12.0.2:Theorem A : A metric space X is compact if and only if X is complete and totally bounded.

Proof: Assume that X is compact. Let$\{x_n\}$.be any Cauchy sequence in X. Since X is sequentially compact.$\{x_n\}$contains a convergent subsequence say $\{x_{nk}\}$.

Let x = lim $\{x_{nk}\}$.We show that $\{x_n\}$ converges to x.

If $\in > 0$ there exist positive integers N_0 and N_{k0}, such that

If $\varepsilon > 0$ there exist positive integer $N_0$and N_{k_0}, such that

$d(x_n, x_m) < \frac{\varepsilon}{2}$ for $n > m \geq N_0$ and $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for $n_k \geq N_{k_0}$
we may choose $N_{k_0} \geq N_0$. We then have for $n \geq N_{k_0}$.
$d(x_n, x) \leq (x_n, X_{n_{k_0}}) + d\left(x_{n_{k_0}}, x\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
Hence $\lim x_n = x$.
Since every Cauchy sequence in X converges in X, X is complete.

To prove that X is totally bounded ·let $\varepsilon > 0$ be any number.The collection $\{S_\varepsilon(x)/x \in X\}$ is an open cover tor X. Since X is compact, there exist finitely many elements of X, say $x_1 \ldots\ldots x_n$ such that $X = \bigcup_{i=1}^{n} S_\varepsilon(x_i)$. This is true for every $\varepsilon > 0$, so X is totally bounded.

Conversely suppose X is complete and totally bounded. Let $\{x_n\}$ be any sequence in X. We show that $\{x_n\}$ has a convergent subsequence. Since X is complete it is enough to show that $\{x_n\}$ has a subsequence which satisfies Cauchy criterion.

Write $x_n = x_{i,n}$ and S_i for $\{x_{i,n}\}$ since X is totally bounded the collection $\left\{S_{\frac{1}{4}}(x)/x \in X\right\}$ has a finite subcollection which covers X. Denote this finite subcollection by $V_1, V_2, \ldots\ldots\ldots\ldots V_N$ Since the elements $x_{i,n}$ belong to the union of V_i, $1 \leq i \leq n$,one of these neighborhoods contains $x_{i,n}$ for infinitely many n.
Let $S_2: \{x_{2.1}, x_{2.2}, \ldots\ldots\ldots x_{2.n} \ldots\ldots\ldots\}$ be such a sequence which is included in a single V_i so that $d(x_{2,i}, x_{2,j}) < \frac{1}{2}$ for all i and j

Apply the above argument to the sequence S_2 and the collection $\left\{S_{\frac{1}{6}}(x)/x \in X\right\}$ has a finite subcollection which covers X.As above we get a subsequence of S_2.

Say $S_3 = \{x_{3.1}, x_{3.2} \ldots\ldots x_{3.n} \ldots\ldots\}$ whose elements lie in one of the spheres so that $d(x_{3.i}, x_{3.j}) < \frac{1}{3} \forall i, j$.
We repeat this process and get a sequence of sequences $\{S_k\}$ where $S_k = \{x_{k.1}, x_{k.2} \ldots\ldots\ldots, x_{k.n} \ldots\ldots\}$ is a subsequence of its predecessor S_{k-1} and $d(xk.i, xk.j) < \frac{1}{k} \forall i, j$.
The diagonal sequence $S = \{y_1, y_2, \ldots\ldots\ldots, y_k,\}$
Where $y_k = x_{k.k}$ $\forall$ k satisfies the conditions SAQ 9.2.
Thus if $r > s$, $d(x_{r.s}, x_{s.s}) < \frac{1}{s}$

If $\varepsilon > 0$ and $s > \frac{1}{\varepsilon}$ then for $r > s$

$$d(y_r, y_s) < \frac{1}{s} < \varepsilon$$

Henee $\{y_r\}$ is a Cauchy sequence and as each $y_r = x_{r.r}$ is an element of $\{x_n\}$, $\{y,\}$ is a Subsequence of $\{x\}$.This completes the proof.

Since a closed subspace of a complete metric space is complete we have the following theorem as an immediate consequence of theorem A.

12.0.3:Theorem B: A closed subspace of a complete metric space is compact if and only if it is totally bounded.

12.0.4:Definition: A subset F of $C(X, \mathbb{C})$ is said to be equieontinuous if for eyery positive number $\in$ there corresponds $\delta(\varepsilon) > 0$ depending on ε such that for every x, y in X with d (x, y) < $\delta(\varepsilon)$ and f $\in$ F such that $|f(x) - f(y)| < \varepsilon$:

12.0.5: Remark: Since every f $\in$ C(X, $\mathbb{C}$) is uniformly continuous given $\in > 0$ and f $\in$ C(X, $\mathbb{C}$) there exists $\delta > 0$ depending on ε as well as f such that $x \in X$, $Y \in X$ and $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

The property that makes a family of functions F in C(X, $\mathbb{C}$) equicontinous, is the existence of a common $\delta(\varepsilon) > 0$ depending on ε alone, such that $d(x,y) < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$ for all f in F.

12.0.6:SAQ: Every finite set $F \subseteq C(X, \mathbb{C})$ is equicontinuous.

12.0.7:SAQ: If (X.d) is any metric space, not necessarily compact, A $\subseteq$ X and if for every δ > 0 there exist finitely many points x_1, x_m in X such that $A \subseteq \cup_{i=1}^{m} S_\delta(x_i)$ then there exist finitely many points a_1..........a_n in A such tlun $A \subseteq \cup_{i=1}^{m} S_\delta(a_i)$

12.0.8:ProPosition: A totally bounded subset $F \subseteq C(X, \mathbb{C})$ is equicoruinuous.

Proof: Since F_1 is totally founded, given $\in > 0$ there exist finitely many elements f_1.........f_n depending upon ε such that $F_1 \subseteq \cup_{i=1}^{m} S_{\frac{\varepsilon}{3}}(f_i)$

Since X is compact and each f_l is continuous on X. corresponding to ε and f_l there exists $\delta_i > 0$ such that $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$ for x, y in X satisfying d(x,y) <δ_i

Let $\delta = \min\{\delta_1 \ldots\ldots \delta_n\}$

If $f \in F_1$ for some i, $f \in S_{\frac{\varepsilon}{3}}(f_i)$ so that $\forall x \in X |f(z) - f_i(z)| < \frac{\varepsilon}{3}$

If d(x, y) < δ and $x \in X$, $y \in X$,then d(x, y) < δ_i for some i so that $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$

Hence $|f(x) - f(y)| = |f(x) - f_i(x) + f_i(x) - f_i(y) + f_i(y) - f(y)|$

$$\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

The proof complete.

12.0.9: SAQ: If $F \subset C(X, K)$ is totally bounded then F is bounded.

12.0.10:SAQ: Let (X. d) be a compact metric space, If $f_n \in C(X, \mathbb{C})\ \forall$ n and $\{f_n\}$ converges uniformly on X then $\{f_n\}$ is equicontinuous on X.

12.1: ASCOLI'STHEOREM:

12.1.1:Theorcm (Ascoli): Suppose F is a closed subset of $C(X, \mathbb{C})$.Then F is compact if and only if F is bounded and equiconrinuous.

Proof: Suppose F is compact totally bounded hence by 12.0.9 equiconrinuous. Moreover a totally bounded set is bounded. Thus compactness of Fimplies that F is bounded *and* equicontinuous.

Conversely suppose that F is bounded and equicontinuous. To prove that F is a compact subset of the metric space $C(X, \mathbb{C})$, it is enough to show that F is sequcntially compact in $C(X,\mathbb{C})$. As we have assumed that F is a closed subset of $C(X,\mathbb{C})$. Complete as a metric space so that every Cauchy sequence in F is convergent. Thus it is enough to show that every sequence in F contains a sub sequence when satisfies Cauchy's criterion for convergence in $C(X,\mathbb{C})$.

Since X is a compact metric space, X is separable. Hence there is a countable set which is dense in X. Let D = $\{x_1.....,x_n......\}$ be any such countable denseset in X.

Since F is bounded, there exists a real number K > 0 such that $|f(x)| \leq K$ for all f in F and x ϵ K.........(1). Since F is equicontinuous, given $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for all x, y in X and f in F. $d(x,y) < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}$. ....(2)

We claim that the open spheres $\{S_\delta(x_n)/n \geq 1\}$ where $\delta = \delta(\varepsilon)$ cover X. Since D is dense in X for any x ϵX the open sphere S_δ (x) contains X_m for some m, so that $X = \bigcup_{m=1}^{\infty} S_\delta(x_m)$

Since X is compact this open cover has a finite subcover, So there are integers $m_1, m_2 \ldots\ldots\ldots m_n$. Such that $X = \bigcup_{m=1}^{\infty} S_\delta(x_{m_i})$

[Text book Print Problem]

By Heine - Borel theorem this sequence of numbers contains a convergent subsequence, Choose any such convergent subsequence say $\{f_{1.1}(x_1), f_{1.2}(x_2).......f_1(x_n)\}$

Write $S_1 = \{f_{1.1}.f_{1.2}.f_{1.3}........f_{1.n}............\}$

Si is a subsequence of So such that $S_1(x_1)=\{f_{1.1}(x_1), f_{1.2}(x_2)\}$ converges,

We define inductively a sequence of sequences $S_n = \{f_{n+1}, f_{n+2},...........f_{n+k}\ldots\ldots\ldots.\}$

Such that for each n, S_n is a subsequence of S_{n-1}, and $S_n(x_n) = \{f_{n-1,1}(x_n), f_{n-2,2}(x_n),...........f_{n-k}(x_n).......\}$ converges; We have already defined such sequence when n = 1. Assuming that S_{n-1}, is already defined.

Then $S_{n-1}(x_n) = \{f_{n-1_1}(x_n), f_{n-1_2}(x_n), \ldots\ldots f_{n-1_k}(x_n) \ldots..\}$ is bounded. Hence contains a convergent subsequence. We choose any such convergent subse quence and denote this by $\{f_{n-1}(x_n), f_{n-2}(x_n), \ldots\ldots f_{n-k}(x_n) \ldots..\}$

we now write $S_n = \{f_{n-1}, f_{n-2}, \ldots\ldots f_{n-k} \ldots..\}$

S_n is a subsequence of S_{n-1} and S_n (x_n) is a convergent sequence. The inductive process is complete. We now apply SAQ12.0.1 to the countable collection $\{S_0, S_1, \ldots\ldots, S_n\ldots\ldots\}$

The sequence $S = \{f_{1.1}, f_{2.2}, \ldots.f_{n.n}.....\}$ is a subsequence or $\{f_1, f_2 \ldots\ldots\ldots f_n, \ldots\ldots\}$

Also for all $k \geq 1$, $\{f_{k_{,k}}, f_{k+1_{,k+1}}, \ldots.. f_{k+p_{k+p}}\}$ is a subsequence of $\{f_{k_{,k}}, f_{k_{,k+1}}, \ldots.. f_{k,p_{k+p}}\}$

Since the sequence S_k (x_k) converges and $\{f_{k_{,k}}(x_k), f_{k_{,k+1}}(x_k), f_{k_{,k+2}}(x_k) \ldots.. f_{k_{,k+p}}(x_k)\}$ is a subsequence of S_k (x_k) this subsequence converges.

Hence $S(x_k) = \{f_{1_1}(x_k), f_{2_2}(x_k), f_{3_3}(x_k) \ldots.. f_{k_k}(x_k)\}$ converges for every k.

Write $g_n = f_{n0}$.Then $\{g_1, g_2.....g_n......\}$ is a subsequence of $\{f_1, f_2, f_3,......,f_n........\}$ and the sequcnce $\{g_n.(x_k)\}$ converges for every k. We show that the sequence $\{g_n\}$ is a Cauchy sequence in F using (2) and (3).

Since $\{g_n(x)\}$ converges for $1 \leq i \leq S$ (s as in (3)) for each i, $1 \leq i \leq s$ ∃ a positive integer N_i such that $|g_n(x_i) - g_m(x_i)| < \frac{\varepsilon}{6}$ for $n \geq m \geq N_i$............(4)

Let N (ε) = max (N_1.............N_s) and $x \in X$.

By (3) ∃ a i ∋ $1 \leq i \leq s$ and $x \in S_\delta(x_i)$

For $n \geq m \geq N(\epsilon)$, $n \geq m \geq N_i$ so

$$|g_n(x) - g_m(x)| = |g_n(x) - g_n(x_i) - g_n(x_i) - g_m(x_i) + g_m(x_i) - g_m(x)|$$
$$\leq |g_n(x) - g_n(x_i)| - |g_m(x_i) - g_n(x_i)| + |g_m(x_i) - g_m(x)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} < \varepsilon \text{ by (4) and (2)}$$

Since this is true for every $x \in X$, we get for $n > m \geq (\varepsilon)$

$d(g_n, g_m) = Sup_{x \in X}|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} < \varepsilon$

Hence $\{g_n\}$ is a Cauchy sequence.

The proof is complete .

12.1.2:Corollary:

Let K be either $\mathbb{R}$ or $\mathbb{C}$, X be a compact metric space and F be a closed subset of C(X.K). Then F is compact if it is equicontinuous and $F_x = \{f(x) / f \in F\}$ is bounded for every $x \in X$.

Proof: In view of Ascoli's theorem it is enough to show that F is bounded in C(X.K) that is there exists a $K > 0$ such that $|f(x)| \leq K$ for $x \in X$ and $f \in F$. Since F is equicontinuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ for all f in F and x, y in X with $d(x, y) < \delta$. The collection of open spheres $\{S_\delta(x) / x \in X\}$ is an open cover for X. So there is a finite number of elements, say, x_1........x_m in X such that $x = \bigcup_{i=1}^{m} S_\delta(x_i)$

Since F_{x_i} is bounded for every i, $1 \leq i \leq m$, there is a $M > 0$ such that. $|f(x_i)| < M$ for every $f \in F$ and $1 \leq i \leq m$. If $x \in X$ there is i such that $d(x. X_i) < \delta$. This implies that $|f(x) - f(x)| < 1$ for evcry $f \in F$

Hence $|f(x)| \leq |f(x_i)| + |f(x) - f(x_i)| < M+1$ ∀ $f \in F$. Since this is true for every $f \in F$ and $x \in X$ it follows that F is bounded.

12.2: MODEL EXAMINATION QUESTION :

1. Define equicontinuity of a family of functions F in $C(X, \mathbb{C})$.when X is a compact metric space. Show that if $F \subseteq C(X, \mathbb{C})$is totally bounded in $C(X, \mathbb{C})$.then F is equicontinuous.
2. Let (X, d) be a compact metric space and $F \subseteq C(X, \mathbb{C})$. If F is compact then F is equicontinuous.
3. Let D be a countable set and $\{f_n\}$ be a sequence of complex valued functions such that $\{f_n(x)\}$ is bounded for every $x \in D$. Show that there is a subsequence $\{g_k\}$ of $\{f_n\}$ such that $\{g_k(x)\}$ converges for every $x \in D$.

12.3: EXERCISE :

1. Show that (0,1) is bounded but not totally bounded.
2. Let: f: R → R be uniformly continuous. Define $f_n(x) = f(nx)$ for $n \geq 1$. Is $\{f_n\}$ an equicontinuous family?
3. Suppose $\{f_n\}$ is an equicontinuous on a compact metric space (X.d) and $\{f_n(x)\}$

converges for every x ∈ X. Show that {f} converges ill C(X, ℂ)

4. Let $f_n(x) = \frac{x^{-}}{x^2+(1-nx)^2}$ $(0 \le x \le 1 \; and \; 1)$

(a) Show that $\lim_n f_n(x) = 0 \; (0 \le x \le 1)$

(b) Show that $|f_n(x)| \le 1 0 \le x \le 1$

(c) Show that $f_n(x)$ is not equicontinuous,

(d) Does equicontinuous imply boundedness?

12.4 SUMMARY:

We learrnt that the metric space and continuity in a metric space. We have proved that A closed subspace of a complete metric space is compact if and only if it is totally bounded.

12.5 TECHINICAL TERMS :

Compact set –Totoal boundedness – conitnuity:

The proof of Ascoli's theorem requires consideration of a countable collection of sequences which, when arranged in a sequence. each one is a subsequence of its predecessor.We recall that a sequence $\{b_n\}$ is a subsequence of a sequence $\{a_n\}$ if there is a strictly increasing map:ϕ:ℕ → ℕ such that $b_n = a_{\phi(n)}$ for every n ≥1.

This definition is equivalent to the existence of a strictly increasing sequence of positive integers (n_k) such that for every k ≥ 1. $b_k = a_K$

Notation: Suppose : X is set and $\{x_1, x_2 \ldots x_n \ldots\}$ is a sequence in X. We write $S_0 = \{x_1, x_2 \ldots x_n \ldots\}$.

Suppose we are given a countable collection of sequences $\{S_0, S_1, S_2 \ldots S_n \ldots\}$ such that each S_k is a subsequence of its predecessor S_{k-1}. We write $S_k = St = \{x_{k+1}, x_{k+2}, \ldots x_{k+n} \ldots\}$.

12.6 ANSWER TO SELF ASSESSMENT QUESTIONS :

SAQ:12.0.1: Let ϕ_{k+1}:ℕ→ ℕ be the strictly increasing map that makes

$S_{K+1}: \{x_{k+1,1}, x_{k+2,2}, \ldots \ldots, x_{k+1,n} \ldots \ldots\}$ subsequence of $S_K = \{x_{k_{,1}}, x_{k_{,2}}, \ldots \ldots, x_{k_{,n}}, \ldots \ldots\}$

Then $x_{k_{,n}} = x_{k+1,\phi(n)}$ for every k ≥1 and n ≥1where $x_{0_{,n}} = x_n$.

We define $\phi_{k_{,p}} : \phi_{k+1}\phi_{k+2} \ldots \ldots \phi_{k+p}$

Then $\phi_{k_{,p}}$=ℕ → ℕ is strictly increasing and $x_{k+p,n} = x_{k+p-1,\phi(n)}, \forall n \ge 1$

Thus $\{x_{k+p,n}\}$ is asubsequence of $\{x_{k,n}\}$ for all p ∈ ℕ. Also if k + p = r.

$\{y_r, y_{r+1}, \ldots y_{r+n}, \ldots\}$ is a subsequence of $\{x_{k,k}, x_{k,k+1}, \ldots x_{k,k+n}, \}$. Which is a subsequence of S_k,.

In particular$\{x_{1,1}, x_{2,2}, \ldots x_{n,n} \ldots\}$ is a subsequence of $S_0 = \{x_1, x_2, \ldots x_n \ldots\}$

12.0.6:SAQ: Suppose $\varepsilon > 0$. There are $\delta j(\varepsilon) > 0$ such that

$|f_j(x) - f_j(y)| < \varepsilon$ if d (x.y) < $\delta j(\varepsilon)$

Set $\delta(\varepsilon) = Min\,\{\delta_i(\varepsilon), \delta_n(\varepsilon)\}$

12.0.7:SAQ: Suppose $\delta > 0$.Then there are $x_1, \ldots\ldots x_m$ in X such that $A \subseteq S_{\delta/2}(x_1) \cup \ldots\ldots \cup S_{\delta/2}(x_m)$

Suppose $a \in A \cap S_{\delta/2}(x_j)$ then $a \in A$ $A \cap S_{\delta/2}(x_j) \subseteq S_\delta(a)$ choose one element from each non-empty set $A \cap S_{\delta/2}(x_j)$

Let $a_1, a_2 \ldots\ldots\ldots a_n$ be the points so selected that $A \subseteq S_{\delta/2}(a_1) \cup \ldots\ldots\ldots \cup S_\delta(a_n)$.

12.0.9.SAQ: 1 > 0, So there are $f_1 \ldots\ldots f_m$ in F such that $F \subseteq S_1(f_1) \cup \ldots . \cup S_1(f_m)$.

Let $K = \|f_1\| + \ldots\ldots + \|f_m\|$. If $f \in F$ then there is j such that $f \in S_1(f_j)$ so

$\|f\| = \|f - f_j + f_j\| \le \|f - f_j\| + \|f_j\| \le K + 1$

12.0.10: SAQ: Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly on X. there is a positive integer N such that $\|f_n - f_m\| = SUP_{x \in X}|f_n(x) - f_m(x)| < \frac{\varepsilon}{3}$ for $n > m \ge N$.

In particular $\|f_n - f_m\| < \frac{\varepsilon}{3}$ for $n \ge N$.

Each of the functions $f_1 \ldots\ldots f_N$ is continuous, hence uniform Jy continuous on X.

Hence there is $\delta > 0$ sud, that $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$ if $d(x, y) < \delta$ and $1 \le i \le N$ If $n \ge N$ and d (x,y) < δ

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

12.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Dr. V. Amarendra Babu

LESSON –13
SEPARATION

OBJECTIVES:

The objectives of this lesson are to.

- ❖ To understand the concepts of a Hausdorff space.
- ❖ To understand the concepts of separation axioms.

STRUCTURE:

13.0: INTRODUCTION :

In this lesson we introduce three separation axioms and explain some of their properties. These axioms are called separation axioms for the reason that they involve separating certain kinds of sets from one another by disjoint open sets.

Consider thae fact that in $\Re$ and $\Re^2$ each one point set is closed. But this is not true in arbitrary topological spaces. For example, consider the topology $I = \{\phi, x, \{a,b\}, \{b,c\}, \{b\}\}$ on the three point set X={a,b,c}. In this space the one point {b} is not closed , for its complement is not open therefore, one often opposes an additional condition that will rule out examples like this one, bringing the class of spaces under consideration closer to those to which one's geometric intuition applies. The condition was suggested by the mathematician Felix Hausdroff. So mathematicians have come to call it by his name.

The Hausdorff condition is stronger than the following property, which is usually called the T_1-axiom.

13.1: HAUSDORFF SPACE:

13.1.1: Definition: A T_1-space is a topological space in which given any pair of distinct points, each has a neighborhood which contain the other.

13.1.2 Examples:

(i) Every discrete space with more than one point is a T_1-space.

(ii) Every indiscrete space with more than one point is not a T_1-space

(iii) Consider the space X = {1,2,3}, T{ ϕ ,x,{1},{1,2},{1,3}} every open set that contains 2 also contains 1.Hence X is not a T_1-space.

(iv) Let X be any infinite set, and let the topology consist of the empty set ϕ together with all subsets of X whose complements are finite (that is co-finite topology) This is a T_1-space.

13.1.3 Self assessment Questions:

Show that any subspace of a T_1 - space is a also a T_1 - space. In the following theorem we will give a simple characterization of a T_1 - space.

13.1.4 Theorem: A Topological space X is a T_1-space if, and only if every sub-set consisting of exactly one point is closed.

Proof: If x and y are district points of X in which every subset consisting of exactly one point is closed, then $\{x\}^1$ is a open set containing y but not x. While $\{y\}^1$ is an open set containing x but not y. Thus X is a T_1 - space.

Conversely, let us suppose that X is a T_1- space and that x is a point of x.
Then by definition 11.1.1 if $y \neq x$ there exists an open set G_y containing y but not x, that is $y \in G_y \subseteq \{x\}^1$.

But then $\{x\}^1 = \cup\{\{y\}: y \neq x\} \subseteq \{x\}^1$, and so $\{x\}^1$ is an union of open sets, andhence is itself open. Thus $\{x\}$ is a closed set for every $x \in X$.

13.1.5: Self Assessment Questions:

Show that in a T_1 - space X, a point x is a limit point of a set E if and only if every open set containing x contains an infinite number of distinct points of E.

13.1.6: Self Assessment Questions:

Show that any finite T_1-space is discrete.

13.1.7: Self Assessment Questions:

Show that a topological space is a T_1-space iff each point of X is intersection ofall open sets containing it. We now define a separation property which is slightlystronger than the T_1- axiom.

13.1.8: Definition: A T_2 - space or Hausdorff space is topological space X in which each pair of distinct points can be separated by open sets, in the sense tha they have disjoint neighborhoods. That is $x \in X$, $y \in X$ and $x \neq y$, there exists neighborhoods U_x, U_y, of X respectively such that $U_x \cap U_y \neq \phi$.

13.1.9: Examples:

(i) Every discrete space X is a T_2-space for, if x, $y \in x$ are such that $x \neq y$,{x} and {y} are open sets, $\{x\} \cap \{y\} = \phi$ and $x \in \{x\}$, $y \in \{y\}$.

(ii) Every metric space is a Hausdorff space.
(iii) Every Subspace of a Hausdroff space is a Hausdroff space.
(iv) Every Hausdroff space is a T_1-space but converse is not true. For example ,if T is the

co-infinite topology on an infinite set X then (X,T) is a T_1- space butnot a Hausdroff space (T_2 - space)

By the definition of T, Since any finite subset of X is closed, singletons areclosed. Hence, (X,T) is a T_1 - space.

We will show that in this space we cannot find two disjoint open sets neitherof which is empty. For otherwise, suppose G and H are disjoint non-empty open sets then, $X = \phi^1 = (G \cap H)^1 = G^1 \cup H^1$, a contradiction, Since G^1 and H^1 are finite so is their union $G^1 \cup H^1 = X$. Therefore (X,T) is not a Hausdroff space.

13.1.10 Theorem: The product of any non empty class of Hausdroff space is a Hausdroff space.

Proof: Let $X = \prod_i X_i$ be the product of nonempty class of Hausdroff spaces. Let x and y be the two distinct points in X. Then we must have $x_{i_0} \neq y_{i_0}$ for at least one index i_0. Since x_{i_0} is Hausdroff, there exists disjoint open sets U_{i_0} and V_{i_0} containing x_{i_0} and y_{i_0} respectively. Now $(\prod_{i_0}^{-1} U_{i_0})$ and $\prod_{i_0}^{-1}(V_{i_0})$ disjoint open sets in the product space containing x and y respectively.

13.1.11: Theorem: In a Hausdorff space any point and disjoint compact subspacecan be separated by open sets in the sense that they have disjoint neighborhoods.

Proof: Let X be a Hausdorff space, x a point in X and C a compact subspace ofX which does not contain x. We exhibit a disjoint pair of open sets G and H suchthat $x \in G$ and $C \subseteq H$. Let y be a point in C. Since $y \neq x$ and X is a Hausdorff space. There exists disjoint neighborhoods G_y and H_y of x and y respectively. If we allow y to vary over C, we obtain a class $\{H_y\}_{y \in C}$ of open sets such that $C \subseteq \bigcup_{y \in C} H_y$ .Since C is compact, there is a finite subclass $\{H_{y_1}, H_{y_2}, \ldots . H_{y_n}\}$such that. $C \subseteq H_{y_1} \cup H_{y_2} \cup \ldots . . \cup H_{y_n}$.

If Gy_1, Gy_2, ... ,Gy_n are the neighborhoods of X which correspond toH_{y_i}'s put $G = \bigcap_{i=1}^{n} Gy_i$ and $H = \bigcup_{i=1}^{n} H_{y_i}$. Clearly G and H are open sets containing x and C respectively. For each i = 1,2,…,n $G \cap H_{y_i} \subseteq G_{y_i} \cap H_{y_i} = \phi$.

Therefore $G \cap H = \bigcup_{i=1}^{n}(G \cap H_{y_i}) = \phi$. Hence, G and H are disjoint. We have provedin theorem 9.1.3 that every closed subspace of a compact space is compact. By considering the indiscrete space X, we have proved that a compact subspace of a compact space X need not be closed. We now use the preceding theorem toshow that compact subspaces of Hausdorff spaces are always closed.

13.1.12: Corollary: Every compact subspace of a Hausdorff space is closed.

Proof: Let C be a compact subspace of a Hausdorff space X. We prove that C isclosed by showing that its complement C^l is open. C^1 is open if it is empty. Sowe may assume mat C^1 is non-empty. Let x be any point in C^1. By theorem to 13.1.1, x has a neighborhood G_X such that $x \in G \subseteq C^1$. Clearly, $C^l = \bigcup_{x \in C^1} G_x$; therefore C^1 is open. One of the most useful consequences of this result is the following:

13.1.13: Theorem: A one–to–one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.

Proof: Let f: X→ Y be a one-to-one continuous mapping of a compact space X onto a Hausdorff space Y. We must show that f(G) is open in Y whenever G isopen in X. If G is open in X then G^1 is closed in X. Since X is compact, G^1 is compact. Therefore $f(G^1)$ is compact since f is continuous. Since f is onto $f(G)^1 = f(G)^1$ is a compact subspace of a Hausdorff space Y. Hence. By Corollary 13.1.12. $f(G)^1$ is closed. Therefore, f(G) is open.

13.1.14: Self Assessment Questions:

(a) Give an example of a topological space in which any sequence converges toevery point of the space.

(b) If X is a Hausdorff space, show that every convergent sequence in X has aunique limit.

13.1.15: Definition: Let X be a topological space: and consider the set C(X, $\mathbb{R}$)of all bounded continuous real functions defined on X. If for each pair of distinct points x and y in X there exists' a function f in C(X, $\mathbb{R}$) such that $f(x) \neq f(y)$, wesay that C(X, $\mathbb{R}$) separates points.

13.1.16: Lemma: If C(X, $\mathbb{R}$) separates points then X is Hausdorff space

Proof: Let x.y$\in$X such that $x \neq y$. Since C(X, $\mathbb{R}$) separates points there exists afunction f in C(X, $\mathbb{R}$) such that $f(x)\neq f(y)$. Suppose f(x)<(y). Let r be a real number such that f(x)<r<f(y}. Now put $G_X = f^{-1}(-\infty, r)$, $G_y = f^{-1}(r, \infty)$ Since f is continuous, G_x and $G_y$are open in X and $x \in G_x$, and $y \in G_y$, $G_X \cap G_y = \phi$ Hence, X is Hausdorff space.

13.1.17:Definition: A topological space X is said to be a completely regularspace if (i) X is a T_1-space (ii) x$\in$X, F is a closed subspace of X such that $x \notin F$ then there exists a function f in C(X, $\mathbb{R}$) such that $0 \leq f(x) \leq 1 \forall x \in X$ and f(x)=0 and f(F)=I. Thus completely regular spaces are T_1-spaces in which continuousfunctions separate points from disjoint closed subspaces.

13.1.18: Lemma: Every completely regular space is a Hausdorff space.

Proof: Let X be a completely regular space, Then X is a T_1 – space by definition.we will show that C(X, $\mathbb{R}$) separates points. Let x,y$\in$X such that $x \neq y$. Since Xis a T_1-space singletons are closed. Thus {y} is closed and $x \notin \{y\}$. Then there exists an f in C(X, $\mathbb{R}$) with values in [0,1] such that f(x)=0 and f(y)=1. Hence for any x,y in X such that $x \neq y$ there exists in C(X, $\mathbb{R}$) such that $f(x) \neq f(y)$. By Lemma 13.1.17. X is a Haussdorff space.

13.1.19: Remark: Any subspace of a completely regular space is completely regular. Our next separation property is similar to that of a Housdorff space, except that it applies to disjoint closed sets instead of merely distinct points.

13.1.20: Definition: A T_1-space X is said to be normal space if for any two disjoint closed sets F_1 and F_2 in X there exist disjoint open sets G_{F1} and G_{F2} such that $F_1 \subseteq G_{F1}$, and $F_2 \subseteq G_{F2}$.

Note: Any metric space is a normal space (see 13.4(5))

13.1.21: Theorem: Every compact Hausdorff space is normal.

Proof: Let X be a compact Hausdorff space and F_1 and F_2 be disjoint closed subsets of X. We must produce a disjoint pair of open sets G_{F_1} and G_{F_2} such that and $F_1 \subseteq G_{F1}$, and $F_2 \subseteq G_{F2}$. If either of the closed sets is empty. We can take the empty set as a neighbourhood of F_1 and the full space as a neighborhood ofthe other. We may therefore assume that both F_1 and F_2 are disjoint compact subspaces of X. Let x be a point of F_1 then $x \notin F_2$ hence by theorem 13.1.11, there exist disjoint open sets G_{F_1} and G_{F_2} , such that $x \in G$ and $F_2 \subseteq G_{F2}$. The collection $\{G_x / x \in F_1\}$ covers F_1, and since F_1, is compact there exist $x_1, x_2 \ldots\ldots\ldots$ in F_1, such that $F_1 \subseteq \bigcup_{i=1}^{n} G_{x_i}$.

Now, put clearly $G_{F_1} = \bigcup_{i=1}^{n} G_{x_i}$, is an open set containing F_1, Put $G_{F_2} = \bigcup_{i=1}^{n} G_{x_i}$

$G_{F_1} \cap G_{y_i} \subseteq G_{F_2} x_i = \phi$. Hence $G_{F_1} \cap G_{F_i} = \phi$ for i= 1,2,......................n Therefore

$G_{F_1} \cap G_{F_2} = (\bigcup_{i=1}^{n} G_{x_i}) \cap G_{F_2} = \bigcup_{i=1}^{n} (G_{x_i} \cap G_{F_i}) = \phi$.

Hence G_{F_1} and G_{F_i} are disjoint open sets such $F_1 \subseteq G_{F_1}$ $F_2 \subseteq G_{F_2}$.

Since $F_2 \subseteq G^{x_i}{}_{F_1}$ for i = 1,2,.........n.

Therefore X is a normal space.

A characterization of normality is given in the following theorem. Let us recallthat by a neighborhood of a set F we mean an open set G containing F.

13.1.22 Theorem: A topological space X is normal if and only if each neighborhood of a closed set F contains the closure of some neighborhood of F.

Proof: Suppose X is normal and the closed set F is contained in an open set G.Put, K = X– G . Now K is a closed set which is disjoint from F. Since X is normalthere exist disjoint open sets G_F and G_K such that $F \subseteq G_F$ and $K \subseteq G_k$. Since $G_F \subseteq X \setminus G_k$ and $X - G_K$ is closed, we have $\overline{G_F} \subseteq X - G_K$.

Now, $\overline{G_F} \subseteq X - G_k \subseteq X - K = G$. Thus G_F is a desired set.

Here G_F is a neighborhood of F and its closure $\overline{G_F} \subseteq G$ Conversely suppose the condition holds and let F_1 be contained in the open set X - F_2, and by hypothesis there exists an open set $G^{\bullet}$ such that $F_1 \subseteq G^{\bullet}$ and $\overline{G^{\bullet}} \subseteq X - F_2$. Clearly $G^{\bullet}$ and X - $\overline{G^{\bullet}}$ form a pair of desjoint open sets containing F_1 and F_2 respectively.

We now prove the main theorem of the lesson that is commonly called the 'Urysohn's Lemma'. It asserts the existence of certain real-valued continuous functions on a normal space X.

13.1.23: Theorem (Urysohn's Lemma) : Let X be a normal space and let A andB be disjoint closed subspaces of X. Then there exists a continuous rea1 funnctionf defined on X, all of whose values lie in the closed unit interval [0,1] such thatf(A) =0 and f(B) =1.

Proof: We shall define, for each rational number r, an open set U_r of X in such a way that whenever r<s we have $\overline{U_r} \subseteq U_s$. For each rational number r such that $r < 0$. define $U_r = \phi$ for each $r > 1$. define $U_r = X$. Let $\{r_n\}$ be a listing of all rational numbers in the interval [0, 1] such that r_1=0 and r_2=1. Define $U_{r_1} = B^1$. Since A is a closed set contained in the open set U_{r_1} by

theorem 13.1.22. there is an open set U_{r_1} r, such that $A \subseteq U_{r_1}$. and $\overline{U_{r_1}} \subseteq U_{r_2}$ suppose that $U_{r_1}, U_{r_2}, \ldots\ldots\ldots, U_{r_n}$. are defined. We define $U_{r_{n-1}}$ as follows: The number r_1 is the smallest element. and r_2 is the largest element of the set $\{r_1, r_2, \ldots.. r_{n+1}\}$ and r_{n+1} is neither r_1 nor r_2 So, r_{n+1} has an immediate predecessor p and an immediate successor q in$\{r_1, r_2, \ldots\ldots\ldots r_{n+1})$ .Since $p < r_{n+1} < q$, sets U_p and U_q are already defined and $\overline{U}_p \subseteq U_q$. Since X is normal, there is an open set $U_{r\ n+1}$ of X Such that $\overline{U}_p \subseteq U_{r_{n+1}}$ and$\overline{U}_{r_{n+1}} \subseteq U_q$. By induction, we have U_{r_n} defined for all n, we now define $f : X \rightarrow \mathbb{R}$ as follows: Given a point x of X, let us define $Q(x) = \{r / x \in U_r\}$, $r<0 \Rightarrow x \notin \phi = U_r$ so, $r \in Q(x) \Rightarrow$ xc. Also Q(x) it contains every number greater thanI, since every x is in U_r, for r > 1. Therefore Q(x) is bounded below, and its greater lower bound is a point in the internal [0,1].
Define f(x) = glb, Q(x) = glb $\{r/x \in U_r\}$.

We show that f is the desired function. If $x \in A$, then $x \in U_r$ for every $r \geq 0$,so that Q(x) equals the set of all.non-negative rationals and f(x) =g.l.b Q(x)=0. similarly, if $x \in B$, then $x \in U_r$ for no $r \leq 1$, so that Q(x) consists of all rational numbers greater than 1, and f(x) =1
We finally we show that f is continuous.
For this purpose, we first prove the following elementary facts.

(i) $x \in \overline{U}_r \Rightarrow f(x) \leq r$

(ii) $x \notin U_r \Rightarrow f(x) \geq r$

To prove (i), note that if $x \in U_r$, then $n \notin U_s$ for every s > r Q(x) contains allrational numbers greater than r, so that by definition we have f(x) -glb $Q(x) \leq r$.

To prove (ii), note that if $x \notin U_r$ then $x \notin U_s$ for any $s \leq r$. Therefore Q(x)contains no rational number less than or equal to r.f(x) = glb $Q(x) \geq r$.
Now we Prove continuity of f. Let $x_0 \in X$. Let (c,d) be an open intervalcontaining the point $f(x_0)$. Choose rational numbers p and q such that $c < p < f(x_0) < q < d$. Put $U = U_q \cap (U_p)'$. Clearly $x_0 \in U$ (for it $x_0 \notin U_q$ then by (ii) $f(x_0) \geq q$. Also $x_0 \notin \overline{U}_p$, because $x_0 \in \overline{U}_p, \Rightarrow f(x_0) \leq p$, by(i)). U is a nbd of x_0. We show that $f(U) \subseteq (c,d)$.
Let $x \in U$ then $x \in U_q \subseteq \overline{U}_q$ so that, $f(x) \leq q$ by (i) And $x_o \notin \overline{U}_p$ po so that $x \notin U_p$ and $f(x) \geq p$ by (ii) Thus $f(x) \in [p, q] \subseteq (c. d)$, as desired.
The following slightly more flexible form of Urysohn's lemma will be usefulin applications.

13.1.24:Theorem : Let X be normal space, and let A and B be disjoint closed subspaces of X. If [a,b] is any closed interval on the real line, then there exists a continuous real function f defined on X, all of whose values lie in [a,b], Such that f(A) = a and f(B) = b

Proof: If a = b, we have only to define f by f(x) = a for-every x, so we may assumethat a < b. If g is a function with the properties stated in Urysohn's lemma, thenthe function f defined by f(x) = (b−a) g(x)+a has the required properties.

13.2: MODEL EXAMINATION QUESTION:

1) Show that a topological space is a T_1-spece if and only if each point is a ,closed set,

2) Show that a one-to-one continuous mapping of a compact space onto aHausdorff

space is a homeomorphism.
3) Define a 'Hausdorff space. Show that every compact subspace of a Hausdorffspace is closed.
4) Define a completely regular space and a normal space. Prove that everycompact Hausdorff space is normal.
5) State and prove Urysohn's lemma:

13.3: EXERCISE :

1) Show that in a T_1-space, no finite set has a limit point.
2) Show that the co-finite topology defined on an infinite set is a T_j-space but not a Hausdorff space..
3) If f is a continuous mapping of a topologica1 space X into a Hausdorff space *Y,* prove that the graph of f={(x,f(x)/x∈X} is a closed subset of the productspace XxY.
4) Show that any metric space is a Hausdorff space .
5) Show that any metric space is a normal space.
6) Show that a closed subspace of a normal space is normal.
7) Let X be a. T_1-space, and show that X is normal iff each neighborhood of aclosed set F contains the closure of some neighborhood of F.
8) Is every normal space a Hausdorff space?
9) 1s a normal space completely regular?
10) Is completely regular space normal?

13.4 SUMMARY:

We learrnt that the Hausdorff space and topological space. We have to prove that the product of any non empty class of Hausdroff space is a Hausdroff space.

13.5 TECHNICAL TERMS:

1. **T1 space**: A topological space where for any two distinct points, there exist open sets containing each point but not the other.
2. **T2 space (Hausdorff space):** A topological space where for any two distinct points, there exist disjoint open sets containing each point.

13.6 ANSWER TO SELF ASSESSMENT QUESTIONS:

13.1.3: Let Y be a subspace of T_I–space X: Let $y_1 \neq y_2$ be distinct elements in Y. Since X is a T_1– space there exists a neighborhood G of y_1 and a neighborhood H of y_2 such that $y_2 \notin G$ and $y_1 \notin H$. Then, G∩Y and H∩ Y are neighborhoods of y_1 , and y_2 in Y. such that $y_1 \notin G\cap Y$ and $y_1 \notin H \cap Y$. Hence, Y is a T_1 – space.

13.1.4: Sufficiency of the condition is obvious. To prove the necessity, supposethere were an open set G containing α for which G∩E was finite.
If we set $G \cap (E/\{x\}) = \bigcup_{i=I}^{\infty}\{x_i\}$ then each set $\{x_i\}$ would also be a closed set.
But then $(\bigcup_{i=I}^{n}\{x_i\})^1 \cap G$ would be an open set containing x with
$[(\bigcup_{i=I}^{n}\{x_i\})^1 \cap G] \cap E \setminus \{x\} = (\bigcup_{i=I}^{n}\{x_i\})^1 \cap (\bigcup_{i=I}^{\infty}\{x_i\}) = \phi$

Thus x would not be a limit point of E.

13.1.16:SAQ: Since X is a T1-space, singletons are closed, Let A be a subset of X then $A = \cup_{a \in A}\{a\}$ is a finite union of closed sets and hence closed. Thus any subset of X is closed and thus any subset of X is open. That is (X, $\mathfrak{I}$) is a discretespace.

13.1.7:SAQ :Let N be the intersection of all open sets containing on arbitrary point x and let y be any point of X different from x. Since the space is T_1, thereexists a neighborhood of x not containing y and consequently y cannot belong to N, that is $y \notin N$. Since y is arbitrary, no point of X other then x can belong to N. It follows that N={x}. Now we prove the converse part let x.y be any two distinct points of X. By hypothesis, the intersection of all neighborhoods of x is {x}, Hence there must be a neighborhood of x which does not contain y. It follows that X is a T_i-space,

13.1.14:SAQ: We first recall the definition of the convergence of a sequence ina topological spcace. Let X be an arbitrary topological space and $\{x_n\}$ a sequenceof points in X. This sequence is said to be convergent if there exists a point x inX such that for each neighborhood G of x a positive integer n_0 can be found withthe property that x_n is in G for all $n \geq n_0$. The point x is called a limit of the sequence, and we say that $\{x_n\}$ converges to x (and symbolize this by $x_n \to x$).

a) **Example :** Consider the indiscrete topological space X consisting of at leasttwo points. This space is not a .Hausdorff space but in this space any sequenceconverges to every point of the space.

Note: This is the reason why the above point x is called a,limit instead of the limit. It is the failure of limits of sequences to be unique that makes this concept unsatisfactory in general topological spaces. The following result shows that this anomalous behavior cannot occur in a Hausdorff' space.

b) In a Hausdorff space, a convergent sequence has a unique limit :- Suppose asequence $\{x_n\}$ converges to two distinct points x and $x^\bullet$ in a Hausdorff space X.Then there exist two disjoint open sets G and $G^\bullet$ such that xeG and $x \in G^\bullet$. Since $x_n \to$ x, there exists a positive integer N 'such that $x_n \in G$ whenever $n > N$. Since $x_n \to x^\bullet$, there exists an integer N such that $x_n \in G^\bullet$ whenever $n > N$. If m is any integer greater than both N and $N^\bullet$, then x_m must be in both G and G', which contradicts the fact that G and $G^\bullet$ are disjoint.

13.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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LESSON-14

TIETZE EXTENSION THEOREM

OBJECTIVES:

The objectives of this lesson are to.

- To understand the concept of normal spaces.
- To prove the Tietze Extension Theorem

STRUCTURE:

14.0 INTRODUCTION :

The Tietze Extension Theorem is a fundamental result in topology, named after the German mathematician Heinrich Tietze. This theorem is a powerful tool for constructing continuous functions on topological spaces. In topology, a continuous function is a function between topological spaces that preserves the topological properties, such as openness and closedness. However, it is not always easy to construct continuous functions on a given topological space. The Tietze Extension Theorem provides a solution to this problem by allowing us to extend a continuous function defined on a subspace to the entire space.

Consider the metric space $X = [0,1]$ and the subset $F = (0,1)$. Since X is a metric space, it is normal, meaning that any two disjoint closed sets can be separated by disjoint open sets.

Define a function $f: F \rightarrow [-1,1]$ by $f(x) = sin(1/x)$. This function is continuous on F, but it cannot be extended to a continuous function $f*: X \rightarrow [-1,1]$.

The reason is that the limit of $f(x)$ as x approaches 0 does not exist. In fact, the function $f(x)$ oscillates between –1 and 1 as x approaches 0.

This example illustrates that even though F is a subspace of a normal space X, a continuous function on F may not be extendable to a continuous function on X.

This highlights the importance of the Tietze Extension Theorem, which provides conditions under which a continuous function on a subspace can be extended to a continuous function on the entire space.

14.1: THE TIETZE EXTENSION THEOREM :

14.1.1: Tietze extension theorem : Let X be a normal space, Y aclosed subspace of X and f a continuous real function defined on Y whose values lie in a closed interval [a,b]. Then f has a continuous extension f^1 defined on all of X whose values also line in [a, b].

Proof: If a = b then f (x)= $a \ \forall x \in Y$ and we define in this case $f^1(x) = a \ \forall x \in X$
Assume that a < b. Since f is bounded. the set {f(x)/x ∈ Y} has l.u.b. M and g.1.b.m.

Since $a \le f(x) \le b$ for every x∈Y we have $a \le m \le M \le b$. we may therefore assume that [a,b] itself is the smallest closed interval such that $a \le f(x) \le b$ for all x∈Y. Since [a,b] is homeomorphic to [–1, 1] we may further assume that $a = -1$ and $b = 1$
Thus f is a continuous function from into [–1,1] and g.l.b {f (x) / x∈Y) = –1and I.u.b {f(x)/x∈Y}=1
Now let $A_1 = f^{-1}\left[-1,\frac{1}{3}\right]$ *and* $B_1 = f^{-1}\left[\frac{1}{3},1\right]$. Then A1, B1 are closed subsets of Y and hence of X. Since –1= g.l.b {f(x)/x∈Y}, there exists a sequence $\{a_n\}$ in Y such that $\lim_n f(a_n) = -1$ similarly there is a sequence $\{b_n\}$ in Y such that $\lim_n f(b_n) = 1$

The implies that A1,B1 are non-empty. Since X is normal by Urysohn's lemma there is a continuous function. G_1: X→ [–1,+1] such that

(a) $g_1(A_1) = -1$
(b) $g_1(B_1) = +1$

Define $f_1 = f - \frac{1}{3}g_1$

Then function f1 is a continuous function $f_1 = Y \to [-\frac{2}{3},\frac{2}{3}]$
To see this first we select $x \in A_1$.
The we have $f_1(x) = f(x) - \frac{1}{3}g_1(x) = f(x) - \left(-\frac{1}{3}\right) = f(x) + \frac{1}{3}$
and $-1 \le f(x) \le -\frac{1}{3}$
there fore $-\frac{2}{3} \le f_1(x) \le 0$
similarly we find for x in $B_1, 0 \le f_1(x) \le \frac{2}{3}$
Now suppose
$x \in Y - (A_1 \cup B_1)$ then $-\frac{1}{3} < f(x) < \frac{1}{3}$
and $-\frac{1}{3} \le \frac{1}{3}g_1(x) \le \frac{1}{3}$ lies between $-\frac{1}{3} + \left(-\frac{1}{3}\right)$ and $\frac{1}{3} + \frac{1}{3}$
i.e. $-\frac{2}{3} \le f_1(x) \le \frac{2}{3}$. Then we note that g.u.b $f_1 = -\frac{2}{3}$, l.u.b $f_1 = \frac{2}{3}$
To see this we note that there is an $n_i \in N$ such that for all $n \ge n_1$
$f(a_n) < -\frac{1}{3}$
This implies that for $n > n_1, a_n \in N$
It follow that
$f = -1$
and hence limt $f_1(a_n) = -\frac{2}{3}$ and g.l.b$_{A_1}$ $f_1 = -\frac{2}{3}$
It follows similarly that $\lim_{n\to\infty}(b_n) = -\frac{2}{3}$ *and* $l.u.b_{B_1} f_1 = \frac{2}{3}$
$A_2 = f_1^{-1}\left[-\frac{2}{3},\frac{2}{3^2}\right] = f_1^{-1}\left[-\frac{2}{3},\frac{1}{3}\left(-\frac{2}{3}\right)\right]$
$B_2 = f_1^{-1}\left[\frac{2}{3^2},\frac{2}{3}\right] = f_1^{-1}\left[\frac{1}{3},\frac{2}{3},\frac{2}{3}\right]$
It is clear that A_2,B_2 are closed subsets of Y and so closed subsets of X
We claim that A_2 is non-empty
We know that

$\lim_{n\to\infty} f_1(a_n) = -\frac{2}{3}$

There fore thus is an $n_2 \in N$ such that for all $n \geq n_2$

$-\frac{2}{3} \leq f_1(a_n) \leq \frac{2}{3^2}$

It follows that for all $n \geq n_2$

$a_n \in A_2$ and $g.l.b_{A_2} f_1 = -\frac{2}{3}$

Similarly

$l.u.b_{B_2} f_1 = \frac{2}{3}$

Since X is normal there is a continuous function

$g_2: X \to \left[-\frac{2}{3}, \frac{2}{3}\right]$

Such that $g_2(A_2) = -\frac{2}{3}$, $g_2 = (B_2) = \frac{2}{3}$

We set

$f_2 = f_1 = \frac{1}{3}\, g_2$. It is checked that

$\lim_{n\to\infty} f_2(a_n) = -\left(\frac{2}{3}\right)^2$, $\lim_{n\to\infty} f_2(b_n) = \left(\frac{2}{3}\right)^2$

inductively we construct a sequence of continuous function on X

g_1……………………, g_{n_1}……………………

such that

$g_m: X \to \left[-\left(\frac{2}{3}\right)^{m-1} \left(\frac{2}{3}\right)^{m-1}\right]$ ……………….(1)

$\left\| f - \frac{1}{3}(g_1 + \cdots + g_m) \right\| \leq \left(\frac{2}{3}\right)^m$ …………….(2)

If we set $f_m = f - \frac{1}{3}(g_1 + \cdots + g_m)$……………(3)

Then

$g.l.b_Y f_m = -\left(\frac{2}{3}\right)^m$ and $l.u.b_Y f_m = \left(\frac{2}{3}\right)^m$

Suppose we have defined g1……….gm having properties 1,2,3 define

$A_{m+1} = f_m^{-1}\left(\left[-\left(\frac{2}{3}\right)^m - \frac{1}{3}\left(\frac{2}{3}\right)^m\right]\right)$

$B_{m+1} = f_m^{-1}\left(\left[\frac{1}{3}\left(\frac{2}{3}\right)^m \left(\frac{2}{3}\right)^m\right]\right)$

Then it is easily checked that

4) $A_{m+1}\, B_{m+1}$ are closed subsets of X

5) $A_{m+1} B_{m+1}$ are non-empty

6) $g.l.b_{A_{m+1}} f_m = -\left(\frac{2}{3}\right)^m$ and $l.u.b_{B_{m+1}} f_m = \left(\frac{2}{3}\right)^m$

By uryshon's lemma there is a continuous function.

$g_{m+1}: X \to \left[-\left(\frac{2}{3}\right)^m \left(\frac{2}{3}\right)^m\right]$

Such that

$g_{m+1}(A_{m+1}) = -\left(\frac{2}{3}\right)^m$ $g_{m+1}(B_{m+1}) = \left(\frac{2}{3}\right)^m$

We find that g1,…….,gm+1

Satisfy conditions (1),(2),(3).

Since

$\|g_m\| \leq \left(\frac{2}{3}\right)^{m+1}$

By weierstars M- test we obtain that the series $\sum_{m=1}^{\infty} g_m$ converges uniformly to a function g:X→ R . As a limit of a uniformly convergent series for functions. $\sum_{m=1}^{\alpha} g_m$.
g is continuous. And we have

$$\left\|f - \frac{1}{3}g\right\| = \left\|f - \frac{1}{3}\lim_n (g_1 + \cdots + g_m)\right\|$$
$$= \lim_m \left\|f - \frac{1}{3}(g_1 + \cdots + g_m)\right\|$$
$$\leq \lim_m \left(\frac{2}{3}\right)^m = 0$$

Thus we have extended f to X; $\frac{1}{3}g$. Is the extension

14.1.2: SAQ : Prove the converse of Tietze extension theorem:

Let X be a topological space. Prove that if every real- valued continuous mappingf of a closed subspace F of X into a closed interval [a,b] can be extended to a continuous real-valued mapping f^* of X into [a.b] then X is normal.

Proof: suppose F_1 and F_2 are two disjoint nonempty closed subsets of X Let [a,b]be any closed interval such that a < b. The mapping f defined by f (x) = a if $x \in F_1$ and f(x) = b if $x \in F_2$ is then a continuous mapping of the closed subs are $F_1 \cup F_2$ into [a.b], Then there exists a continuous function f* of X into [a.b] such that $f^*/ F_1 \cup F_2 = f$. If c is any real number such that a < c < b then f^{*-1} ([a,c]) and f^{*-1} ([c.b]) are disjoint open sets containing F_1 and F_2 respectively. Thus X is normal.

The following example shows that the closedness of F is essential in theabove the theorem.

14.1.3: Example:

Let X = [0,1] and F – (0. 1). Since X is a metric space, it is normal. F is not closed in X. Define. f: F→[–1,1] by f(x) = $\sin\left(\frac{1}{x}\right)$. Then f is continuous. Since $\lim_{x \to \infty} f(x)$ does not exist f can not be extended to a continuous mapping f*of X into [–1. 1].

We now turn our attention on the metrization problem. We begin with thefollowing example.

14.1.4:Example: The Infinite dimensional unitary space c^∞ consisting of all sequences *of* complex $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ is also denoted by I^2 and is a complete metric space with respect to the metric defined by $d((x_n)(y_n)) = \{\sum_{n=1}^{\infty}(x_n - y_n{}^2)\}^{\frac{1}{2}}$

We denote the topology induced by this metric by T_d .Clearly $1^2 \subset \mathbb{C}^n$ where is the space of all sequence of complex numbers.

Since $\mathbb{C}^n = \prod_{n \in N} \mathbb{C}^n$ where $\mathbb{C}_N = \mathbb{C}\ \forall n, \mathbb{C}^N$ has the product topology on it where $\mathbb{C}_n$ is topology equipped with the usual topology induced by the metric $d_n(z_1, z_2) = |z_1 - z_2|\ \forall n \geq 1$ and $z_1 \in \mathbb{C} z_2 \in \mathbb{C}$ we denote the restriction of the product topology on $\mathbb{C}^N$ to I^2 by T and prove that $T \subset_{\neq} T_d$.

For $k \in N$ *and* $\delta > 0$ *write* $S(k, \delta) = \{z = (z_n) \in \mathbb{C}^n : |z_K| < \delta\}$ these sets S(k, δ) are the typical subbasic open sets of d that contain $0 \in \mathbb{C}^N$

Let $k_1 < \ldots\ldots\ldots < k_r$ be a sequence of natural numbers and let $\delta_1, \delta_2, \ldots\ldots\ldots, \delta_r$ be a sequence of positive numbers. The sets

$$S(k, \delta) = S(k_1, k_2 \ldots k_r, \delta_1, \delta_2, \ldots\ldots\ldots, \delta_r)$$
$$= S(k_1, \delta_1) \cap \ldots\ldots\ldots \cap S(k_r, \delta_r)$$
$$= \{(Z_n) \in \mathbb{C}^N : |Z_{k_1}| < \delta_1 \ldots |Z_{k_r}| < \delta_r\}$$

are the typical basic open sets of $\mathbb{C}^N$ that contain $O \in \mathbb{C}^N$
if $z \in I^2$ and $\|z\| < \delta$ then
$|z_K| \leq \|z\| < \delta$
and so
$S_\delta(O) \subset S(k,\delta)$
This implies $T \subseteq T_d$. We claim that there are no sequences $k_1,k_2,\ldots\ldots k_r$ of natural numbers and $\delta_1, \delta_2,\ldots\ldots,\delta_r$ positive numbers such that $S(k,\delta) \subseteq S_\delta(O)$
To see this we set $\delta_o = Min\{\delta_1 \ldots \delta_r\}$
Choose any positive integer k. Take any sequence $Z(k,\delta_0) = \{z_1, \ldots z_K, 0.0 \ldots \ldots 0\}$
$|z_n| < \delta_0$.
The sequence is in I2. This sequence is also in $S(k,\delta)$. However if we take $Z_n = \frac{1}{2}\delta_0$
$n = 1,2 \ldots \ldots, k$
we have
$\|Z(k,\delta_0)\| = \frac{k}{2}\delta_0$
if $\frac{k}{2}\delta_0 > i.e\ k > \frac{2\delta}{\delta_0}$
$\|Z(k,\delta_0)\| > \delta$
and so $Z(k,\delta_0) \notin S(k,\delta)$

The following basic fact about product topology will be in Uryshon's embedding theorem.

14.1.5: Proposition :

Suppose that I is a set, for each in I, (A_i,T_i) is a topological space and (A,T) is the product of the (A,T). Let P_i: $A \to A_f$ be the projection of A onto the i^{th} factor $P_i((x_f)) = x_i$. Then

(1) P_i is continuous ; (2) P_i is a open map and

(3) If Y is a topological space and f : $Y \to A$ is any map, then f is continuous iffp_i of is continuous for all in f in I.

Proof : Let us recall that for $i \in I$, I(J) = 1−{J},
J_{11}.........., $J_m \in I, I(J_1,J_2,\ldots\ldots,J_m)$ and for $V_h,\ldots\ldots V_{j_n}\ldots\ldots V_{j_n}$ in $A_{j_2}\ldots\ldots A_{j_n}$
the sets $V_{j_1} \times \ldots\ldots \times V_{j_n} \times \prod_{i\in I(J_1 \ldots J_n)} A_i$ are basic open sets for T

1) Suppose V_J is an open sets A_J Then $P_j^{-1}(V_j) = V_1 \prod_{i\in I(J)} A_i$ is a sub basic open set of T. This implies that P_j is continuous.
2) It is enough to prove $P_J(V)$ is an open subset for every basic open subset Vof T. Let $V = V_{J_1} \times \ldots\ldots \times V_{J_n} \times \prod_{i\in I(J_1 \ldots J_n)} A_i$ then $P_J V = \begin{pmatrix} V_{jr} & if\ J = J_r \\ A_j & if\ j \notin \{j_r \ldots j_n\} \end{pmatrix}$ and so 2 is proved
3) Consider $W_j \subseteq A_j$ and $(p_j, f)^{-1}(W_j)$
we have
$(p_j, f)^{-1}(W_j) = f^{-1}(W_j \times \prod_{i\in(j)} A_j)$
Therefore p_γ of is continous for all j implies.
$f^{-1}(V)$

is an open set of Y for all sub basic open sets V of A. This implies that f is continuous. Then rest is clear.

we recall a definition.

14.1.6: Definition :

Suppose (X,T) is a topological apace we say that (X,t) is metrizable.

If there is a metric d on X such that the topology T_d induced by the metric don X is the same as T : $T=T_d$

14.1.7: Proposition : Let $J_n = \left[0, \frac{1}{2^n}\right]$ be given the usual topology and let $J = \prod_{n\in N} J_n$ be the product space. with the product topology T define $d(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|$ where x = (x_n), y = (y_n) are elements of J. Then d is a metric on J and thetopology T_d induced by d is the same as T.

Proof : We leave the proof that d is a metric to the reader. In the case of I^2 wehave just proved that $T \subseteq T_d$

The same method will give us in this case also $T \subseteq T_d$

We shall now prove $T_d \subseteq T$

To get a clear idea we shall first prove that the open sphere $S_\delta(0)$.

contains a basic open neighborhood V of O with respect to the producttopology.

We get

$$\delta_0 = Min\{1, \delta\}$$

We can find a positive integer n_0 such that $\frac{1}{n_0} < \frac{\delta_0}{2}$

We set $V = \prod_{k=1}^{n_0} \left[0, \frac{\delta_0}{2n_0} \cdot \frac{1}{2^k}\right] \times \prod_{k=n_0+1}^{\infty} J_k$

For any x = (x_0) in V we have $d(x, 0) = \sum x_n$

$$= \sum_{k=1}^{n_0} x_K + \sum_{k=n_0+1}^{\infty} x_K$$

$$< \sum_{K=1}^{n_0} \frac{\delta}{2n_0} \cdot \frac{1}{2^K} + \sum_{K=n_0+1}^{\infty} \frac{1}{2^K}$$

$$< \frac{\delta_0}{2} + \frac{1}{2^{n_0}}$$

$$< \frac{\delta_0}{2} \cdot \frac{\delta_0}{2}$$

$$= \delta_0$$

Thus $V \subseteq S_\delta(O)$. Suppose now $\alpha = (\alpha_n) \in J$ and we are given $S_\delta(\alpha)$. As above we get $\delta_0 = min\{1, \delta\}$ and choose an n_0 such that $\frac{1}{n_0} < \frac{\delta_0}{2}$. If $m \le n_0$ then we choose intervals as follows

$$I_m = \left(\alpha_m - \frac{\delta_0}{2n_0}\frac{1}{2^m} \cdot \alpha_m + \frac{\delta_0}{2n_0}\frac{1}{2^m}\right) \cap J_m$$

This is an open interval of J_m and

$$V = \prod_{m=1}^{n_0} I_m \times \prod_{k=n_0+1}^{\infty} J_k$$

is a basic open set containing α suppose $x \in V$. Then

$$\sum |x_K - \alpha_K| = \sum_{K=1}^{n_0} |x_K - \alpha_K| + \sum_{K=n_0+1}^{\infty} |x_K - \alpha_K|$$

$$< \frac{\delta}{2n_0} \sum_{K=1}^{n_0} \frac{1}{2^K} + \sum_{K=n_0+1}^{\infty} \frac{1}{2^K}$$

$$< \frac{\delta_0}{2} \cdot \frac{\delta_0}{2} = \delta_0$$

Thus $V \subseteq S_\delta(\alpha)$ this implies that every point α of an open set U with respect to the metric technology T_d has a neighborhood $V(\alpha)$ with respect to the product topology T. So we have U$\in T$ i.e: $T_d \subseteq T$.

Thus the proposition is proved.

14.1.8: SAQ: Let X be a topological space and F a closed subspace of X. Suppose every real-valued continuous mapping f of F into [0,1] can be extended to a continuous real-valued mapping f* of X into [0,1]. Prove that X is normal.

14.1.9: SAQ: Let X be a topological space and F a closed subspace of X. Suppose every real-valued continuous mapping f of F into [-1,1] can be extended to a continuous real-valued mapping f* of X into [-1,1]. Prove that X is normal.

14.2: MODEL EXAMINATION QUESTIONS:

1. State, and prove Tietze extension theorem.
2. Let X be a normal space, Y aclosed subspace of X and f a continuous real function defined on Y whose values liein a closed interval [a,b]. Then f has a continuous extension f^1 defined on all of X whose values also line in [a, b].

14.3: EXCERCISE:

1) Prove that a separable, metric space is second countable
2) Show that every metric space is normal
3) Does the continuous function $\frac{1}{x}$ defined on R–{O} have a continuous extention to the whole of R ?
4) Let X=[−1,1] , Y = $\left[-\frac{1}{2}, \frac{1}{2}\right]$ Define f, f_1, f_2 by

$$f(x) = |x| \; for \; x \in Y$$

$$f_1(x) = |x| \; for \; x \in X \; and$$

$$f_2(x) = \begin{cases} |x| \; if \; x \in Y \; and \\ \frac{1}{2} \; if \; x \in X - Y \end{cases}$$ show that f_1 and f_2 are continuous extension of f.

5) Define f(x) = x sin $\frac{1}{x}$ for x ∈ (0,1] show that f has a unique continuousextension on [0,1].

14.4 SUMMARY:

We learnt that normal space, metric space and continuous extension. We have proved that Tietze extension theorem.

14.5 TECHNICAL TERMS:

1. **Normal Space:** A topological space X that satisfies the separation axiom T_4, meaning that any two disjoint closed sets can be separated by disjoint open sets.
2. **Closed Subspace**: A subspace Y of a topological space X that is closed in the topology of X.

14.6 ANSWER TO SELF ASSESSMENT QUESTIONS:

14.1.8: SAQ: Let A and B be two disjoint closed sets in X. We need to find disjoint open sets U and V in X such that A ⊆ U and B ⊆ V. Define a function f: A ∪ B → [0,1] by f(x) = 0 if x ∈ A and f(x) = 1 if x ∈ B. This function is continuous on A ∪ B. By the given condition, there exists a continuous function f*: X → [0,1] that extends f. Let U = $f\_^{-1}([0,1/2))$ and V = $f\_^{-1}((1/2,1])$. Then U and V are disjoint open sets in X such that A ⊆ U and B ⊆ V.

Therefore, X is normal.

14.1.9: SAQ: Let A and B be two disjoint closed sets in X. We need to find disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. Define a function $f: A \cup B \rightarrow [-1,1]$ by $f(x) = -1$ if $x \in A$ and $f(x) = 1$ if $x \in B$. This function is continuous on $A \cup B$. By the given condition, there exists a continuous function $f^*: X \rightarrow [-1,1]$ that extends f. Let $U = f\_^{-1}([-1,0))$ and $V = f\_^{-1}((0,1])$. Then U and V are disjoint open sets in X such that $A \subseteq U$ and $B \subseteq V$.
Therefore, X is normal.

14.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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Lesson – 15
URYSHON'S EMBEDING THEOREM

OBJECTIVES:

The objectives of this lesson are to.
- ❖ To understand the concept of normality and second-countability.
- ❖ To prove the Urysohn Imbedding Theorem.

STRUCTURE:

15.0 INTRODUCTION:

Urysohn's Imbedding Theorem is a fundamental result in topology that establishes a connection between normality, second countability, and metrizability of topological spaces. In this lesson the theorem provides a sufficient condition for a topological space to be metrizable, which is essential in various applications. The theorem shows that normality and second countability imply metrizability, highlighting the importance of these properties.The theorem enables the imbedding of topological spaces into metric spaces.
Consider the unit square X = [0,1] × [0,1] in $\mathbb{R}^2$, equipped with the subspace topology inherited from the standard topology on $\mathbb{R}^2$.

This space X is:

1. Normal: Since $\mathbb{R}^2$ is a metric space, it is normal. As a subspace of $\mathbb{R}^2$, X inherits this property.
2. Second-countable: The standard topology on $\mathbb{R}^2$ is second-countable, and X inherits a countable basis from this topology.
By the theorem, X is metrizable. In fact, the standard metric on $\mathbb{R}^2$, restricted to X, induces the subspace topology on X.This example illustrates the theorem's statement, showing that a normal and second-countable space (the unit square X) is indeed metrizable.

15.1: THEOREM : (URYSHON IMBEDDING THEOREM)

Suppose X is a topological space. which is normal and secondcountable. Then X is metrizable.

Proof : Let $\{U_n\}_{n\in N}$
be a basis for for the open sets of the topology of X. We consider the ordered pairs of $(m, n) \in N \times N$ of natural numbers such that $U_m \subseteq \bar{U}_m \subset_{\neq} U_n$.
we have assumed that X is normal, By Urysohn's lemma there is a continuous function.
$f_{m,n}: X \to [0,1] \subseteq R$ such that

1) $f_{mn} = 1$ on $\cup_m$ and
2) $f_{mn} = 0$ $outside$ $\cup_n$

The set of ordered pairs (m, n) we have considered is a subset of NxN which is a countable .So the set functions is a countable set. We write them in a sequence { $f_1, f_2, \ldots, f_n, \ldots\ldots$} corresponding to each p ∈ N there is a unique ordered pair (m, n) such that $f_{mn} = f_p$ and conversely. We define a map $F: X \to J$ by

$$F(x) = \left(\frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \ldots \ldots, \frac{f_p(x)}{2^p}, \ldots \ldots\right)$$

we recall that $p_k: J \to J_k$

is the projection map and that $F: X \to J$ is continous if and only if $p_j F = \frac{f_j}{2^j}$

is a continuous function on X. If follows that F is a continuous map into theproduct J of topology spaces J_k.

We denote F(X) by Y. We consider Y with the topology T_o induced by the product topology T on J. We have just proved that $F: X \to Y$ is a continuous onto map.

We will now prove that F is one - to - one and open.

F is one–to–one : Let $x, y \in X$ and $x \neq y$. Then there is a basic neighborhood $\cup_n$ of x, such that $x \in \cup_n \subseteq \bar{\cup}_m$ and $y \notin \cup_n$

Since X is normal we may choose a basic neighborhood $\cup_m$ of x such that $x \in \cup_m \subseteq \bar{\cup}_m \subset \cup_n$

Then we have a p = p(m, n) such that $f_p(x)=1$ and $f_p(y)=0$

This implies that $F(x) \neq F(y)$

$F: X \to Y$ is an open Map : It is enough to show that the image of each $\cup_n$ is an open subset of Y Let $v \in F(\cup_n)$

We shall prove that there is a neighborhood V of v with respect to thetopology T on Y induced by T on J such that $V \subseteq F(\cup_n)$

Since $v \in F(\cup_n)$ there is an u in $\cup_n$ such that v = F(u)

The space X is normal and $u \in \cup_n$.Therefore there is a basic open set $\cup_m$ such that $x \in \cup_m \subseteq \bar{\cup}_m \subset \cup_n$ corresponding to this pair (m, n) of natural numbers there is a p = p(m, n) such that $f_p = f_{mn}$.

we have $f_p(x) = 0$ if $x \notin \cup_n$

The set $V = \left\{z \in J: z_p > \frac{1}{2^{p+1}}\right\}$

is a sub basic open set of J since $f_p(u) = 1$. The point v = F(U)∈V. If x in X satisfies F(x)∈V then we must have

$$\frac{1}{2^p} f_p(x) > \frac{1}{2^{p+1}}$$

that is $\frac{1}{2} < f_p(x)$

this implies that $x \in \cup_n$.

Therefore we have proved that $V \cap Y \subseteq F(\cup_n)$

That is F (∪ n) is an open subset of Y. Since F is continuous and also one - to-one it follows that $F: X \to (Y, T^1)$ is a homeomorphism

In the previous result we have proved that

$T = T_d$ on J. Therefore T on Y is the same as the topology Induced by themetric on Y. Thus we have, proved that X Is homeomorphic to a metrizable space. So X is metrizable.

15.1.1: SAQ: Show that a compact hausdorff apace Is metrizable it is secondcountable

Proof: Let X be compact Hausdorff space. Then it is normal Suppose X issecond countable. Then by urysohn Imbedding theorem, X is metrizable. Conversely suppose that X is metrizable. By SAQ 11.1.13 since X is a compact metric space, It is separable. Since every separable metric space is second countable it follows that X is second countable.

Remark : Theorem 15.1.1 provides an example of a metrizable space . In this Theorem we show that the Tychonoff product of metric spaces is metrizble.In an example we show that the metric topology of the infinite dimensional Euclidean space $\mathbb{C}^\infty$ (also denoted by 1^2) is stronger than the relative topology inherited from the Tychnoff product $\mathbb{C}^N$

15.1.2:Theorem: Suppose for each natural number n, (A_n, d_n) is a metric spaceand T_n. is the topology induced on A_n by d_n. Then the product space (A, T) where $A = \prod_{n\in N} A_n$ and T is the product topology, is metrizable.

Proof: Since the metric $\frac{d_n}{1+d_n}$ and dn generate the same topology on An. we may assume that $0 \le d_n(x,y) \le 1$ for all x,y in A_n.
For x= (x_n) and y = (y_n) in A define $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n,y_n)$
Since $0 \le d_n(x,y) \le 1$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent
The series on the right hand side converges, hence the above definition ismeaningful.
For x = (x_n). y =(y_n) and z = (z_n) in A we have

i. $d(x,y) = 0 \Leftrightarrow d_n(x_n,y_n) = 0\ \forall n \Leftrightarrow x_n = y_n \forall n \Leftrightarrow x = y$

ii. $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}(x_n,y_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n,y_n) = d(y,x)$ and

iii. $d(x,z) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n,z_n)$

$$\le \sum_{n=1}^{\infty} \frac{1}{2^n}\{d_n(x_n,y_n) + d_n(x_n,z_n)\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n}\{d_n(x_n,y_n)\} + \sum_{n=1}^{\infty} \frac{1}{2^n}\{d_n(x_n,z_n)\}$$

Thus d is a metric on A. We denote the topology on A Induced by d by T^1.

What we prove is That $T = T^1$.
For this it is enough to prove that

1) given an open set V of T and $\alpha \in V$ there is a neighborhood $U^1(\alpha)$ of α with respect to T^1 such that $U^1(2) \subseteq V$ and
2) given an open set U^1 in T^1 and $\beta \in U$ there is a neighborhood $V(\alpha)$ of α with respect to T such that $V(\alpha) \subseteq U^1$.

Further it is clear that the statement (1) if proved for a class of sub basic open sets implies the statement for all open sets of T. So it is enough to prove (1)and (2) *for* sub basic open sets V and U.
Let V be a sub basic open set with respect to the product topology T. Then there is a natural number m and an open set V (m) of A_m such that
$V = V(m) \times \prod_{\substack{n\in N \\ n\ne m}} A_n$
Let $\alpha = (\alpha_n) \in V$. Then $\alpha_m \in V(m)$ since T_m is the topology induced by the metric d_m there is a $\delta > 0$ such that $S_\delta(\alpha_m) \subseteq V(m)$
then we claim that the sphere of radius $\frac{\delta}{2^m}$ centered at α.
With respect to d is contained in V: $S_{\frac{\delta}{2^m}}(\alpha) \subseteq V$ Suppose $x \in S_{\frac{\delta}{2^m}}(\alpha)$.

Then we have

$$\frac{1}{2^m} d(x_m.\alpha_m) \le d(x,\alpha)$$

$< \frac{\delta}{2^m}$

This implies that $x_m \in S_\delta(\alpha_m)$

and so $x \in V$. We have, provide (1). Let U^1 be an open set in the topology generated by the metric d on A and let $\beta \in U^1$. By the definition of the topology T^1 there is $\delta > 0$ T^1 such that $S_\delta(\beta) \subseteq U^1$

we choose a natural number k such that

$$\frac{1}{2^k} < \frac{\delta}{2} \text{ i.e } 2^{k-1} > \frac{1}{\delta}$$

This is possible because of Archimedian property of R. we claim that

$$V(\beta) = \prod_{k=1}^{k} S_{\frac{1}{2^k}}(\beta_r) \times \prod_{n>k} A_n$$

is contained in $S_\delta(\beta)$ suppose $y \in V(\beta)$ then

$$d(y,\beta) = \sum_{r=1}^{k} \frac{1}{2^r} d_r(y_r,\beta_r) + \sum_{n=k+1}^{\infty} \frac{1}{2^n} d_n(y_n,\beta_n)$$

$$\leq \sum_{r=1}^{k} \frac{1}{2^r} d_r(y_r,\beta_r) + \frac{1}{2^k}$$

$$\leq \sum_{r=1}^{k} \frac{1}{2^r} \cdot \frac{1}{2^k} + \frac{1}{2^k}$$

$$< \frac{1}{2^k} + \frac{1}{2^k} + \frac{1}{2^{k-1}} < \delta$$

Therefore

$V(\beta) \subseteq S_\delta(\beta) \subseteq U^1$ and We have proved (2)

The result is proved.

15.1.3:SAQ: Suppose X is a topological space that is normal and second-countable. Prove that there exists a homeomorphism between X and a subspace of the Hilbert cube.

15.1.4:SAQ: Let X be a normal and second-countable topological space. Prove that X can be imbedded into a compact metric space.

15.2: MODEL EXAMINATION QUESTIONS:

1. State and prove UrySohn's imbedding theorem
2. Show that the product topology of a countable collection of metric space is metrizable.

15.3: EXCERCISE :

1) Show that a second countable normal space is metrizable
2) Let $I_x = [0, 1]$ $((\forall x \in [0,1]))$ equipped with the usual topology, Show that product topology an $\prod_{x=[0,1]} I_x$ is normal but not metrizable.
3) Given an example of a metric space which is not second countable.

15.4 SUMMARY:

We learnt that the countable, product topology and imbedding. We proved that Urysohn's imbedding theorem.

15.5 TECHNICAL TERMS:

1. Normal Space: A topological space X that satisfies the separation axiom T_4, meaning that any two disjoint closed sets can be separated by disjoint open sets.

2. Second-Countable Space: A topological space X that has a countable basis for its topology.

3. Metrizable Space: A topological space X that can be endowed with a metric that induces its topology.

4. Imbedding: A homeomorphism between a topological space X and a subspace of another topological space Y.

15.6 ANSWER TO SELF ASSESSMENT QUESTIONS:

15.1.3: SAQ: Let X be a normal and second-countable topological space. We need to prove that there exists a homeomorphism between X and a subspace of the Hilbert cube. Since X is second-countable, it has a countable basis $\{U_n\}$. For each n, define a continuous function $f_n: X \to [0,1]$ such that $f_n(x) = 0$ if $x \notin U_n$ and $f_n(x) = 1$ if $x \in U_n$. Define a function $f: X \to [0,1]^\omega$ by $f(x) = (f_1(x), f_2(x), ...)$. Since each f_n is continuous, f is also continuous. Moreover, f is injective, since if $x \neq y$, then there exists n such that $x \in U_n$ and $y \notin U_n$, so $f_n(x) \neq f_n(y)$. Finally, f is a homeomorphism onto its image, since it is continuous and injective, and its inverse is also continuous.Therefore, there exists a homeomorphism between X and a subspace of the Hilbert cube.

15.1.4: SAQ: Let X be a normal and second-countable topological space. Since X is second-countable, it has a countable basis $\{U_n\}$. For each n, define a continuous function $f_n: X \to [0,1]$ such that $f_n(x) = 0$ if $x \notin U_n$ and $f_n(x) = 1$ if $x \in U_n$. Define a function $f: X \to [0,1]^\omega$ by $f(x) = (f_1(x), f_2(x), ...)$. Since each f_n is continuous, f is also continuous. Moreover, f is injective, since if $x \neq y$, then there exists n such that $x \in U_n$ and $y \notin U_n$, so $f_n(x) \neq f_n(y)$. The Hilbert cube $[0,1]^\omega$ is a compact metric space. Therefore, f(X) is a subspace of a compact metric space.
Hence, X can be imbedded into a compact metric space.

15.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Prof. B . Satyanarayana

LESSON – 16

CONNECTEDNESS

OBJECTIVES:

The objectives of this lesson are to.

- ❖ To understand the concepts of connectedness in a topological space.
- ❖ To understand the concepts of a connected topological space.
- ❖ To understand the concepts of discrete and indiscrete topology.
- ❖ To understand the concepts of a continuous function between topological spaces.

STRUCTURE:

16.0: INTRODUCTION:

In this lesson we study connected topological spaces. which is one of the most important topics in topology. Intuitively; a connected space may be thoughtof a space consisting of a single piece. We give a formal definition of a connected topological space. We prove that a subspace of the real line R is connected if, an only if. it is an interval. We also prove that the property of connectedness is preserved by continuous functions. We further prove that the product of a non-empty class of connected spaces is connected and hence R^n and C^n areconnected. We also introduce the concept of components of a topological space. We study some elementary properties of components.

16.1: CONNECTED SPACES:

16.1.1: Definition: A topological space (X. T) is said to be connected if X can not be represented as the union of two non-empty disjoint open sets. In other words; if $X = A \cup B, A, B \in T, A \neq \phi,\ A \cap B \neq \phi$ implies B = ϕ . then X is said to be connected.

16.1.2: Definition: Let (X. T). be a topological space. If there exists A. B in T such that $X = A \cup B, A \neq \phi, B \neq \phi$ and $A \cap B = \phi$ then this representation of X is called a disconnection of X. If X is not connected we say that X is disconnected or equivalently X is disconnected if and only if, X has a disconnection.

16.1.3: Definition: A subspace Y of a topological spade X is said to be connected if Y is connected with respect to the relative (induced) topology in Y.

16.1.4: Lemma:A subspace Y of a topological space X is connected if and only if, Y is not contained in the union of two open subsets of X whose intersectionswith Y are non-empty and disjoint.

Proof. Suppose that Y is connected. Let $Y \subseteq A \cup B$ when: A and B are open in X. Let C= $A \cap Y$,D = B $\cap$ Y . Then Y= C$\cup$D. C and D are open in Y. If C $\cap$ D = ϕ and Then D = ϕ since Y is connected.

Conversely suppose that the stated condition holds. Let Y = C $\cup$D whereC and D are disjoint open sets in Y. Let C = Y$\cap$A , D = Y$\cap$ B where A and B are open in X. Then $Y \subseteq A \cup B$. If $c \neq \phi$ then A$\cap$Y $\neq \phi$. i.e. $B \cap Y = \phi$ D = ϕ . Hence Y is connected

16.1.5: SAQ : Let X be any non-empty set. Let T be the indiscrete topology on X. Show that (X. T) is a connected topological space.

16.1.6: SAQ: Let X be a set with at least two elements. Let T be the discrete topology on X. Show that (X,T) is disconnected.

16.1.7: Lemma: let X = { a. b },Y = [c, d]. Let T_X be the discrete topology on X and let T_y be the discrete topology on Y. Then (X.T_x) and (Y.T_y) are homeomorphic. Thus there exists a unique discrete space with two points upto isometry.

Proof. T_X : { ϕ : {a } { b}. X }. *Ty* = { ϕ .{c}, {d }. Y }.

Define the mapping f: X $\to$Y by

f {a) = c and f (b) = d. Then f is a bijection which is continuous and open. Thus f is a homeomorphism.

16.1.8: Notation: let 0 and 1 be two symbols. The discrete two point space is denoted by {0,1}.

16.1.9: Theorem :A topological space X is disconnected if and only if there exists continuous function from X onto the discrete two point space {0,1}.

Proof: Suppose that X is disconnected and Let X = A$\cup$B be a disconnection of X. Then A and Bare non-empty disjoint open subsets of X.

Define $f: X \to \{0,1\}\ by\ f(x) = \begin{cases} 0\ if\ x \in A \\ 1\ if\ x \in B \end{cases}$

Then f^{-1} (ϕ) = ϕ f^{-1} ({0}) = A. f^{-1}({1}) = B. and f^{-1} ({0.1}) = X and all these sets are open in X. Hence f is continuous.

Also f is onto. since A $\neq$ ϕ , and B $\neq$ ϕ

Conversely suppose that there exists a continuous surjective· function f: X$\to$ {0.1} Let A= {x$\in$X/f(x)=0} and B: {x$\in$X/f(x)=1}. A and B are non-empty, since f is surjective. Also A $\cap$ B = ϕ ,{0} and {1} are open and A = f^{-1} ({0}) = B = f^{-1} ({1}). Since f is continuous. we have that A and B are open in X. Thus X=A$\cup$ B is a disconnection of X. Thus X is disconnected

16.1.10: Theorem: Let f: X $\to$ Y be a continuous mapping of a connected topological space X into a topological space Y. Let Z = f (X) be the (continuous)image of X. Then Z is connected.

Proof: If Z is not connected, then by 16.1.9there exists a continuous functiong from Z onto the discrete two point space {o, I}. Then the mapping h : X$\to$ {0,1} defined by h(x)=g(f(x)). Being the composite of two continuous functions, is continuous and it is also onto. This implies that X is not connected. which is a contradiction to the hypothesis. Hence Z is connected.

16.1.11: SAQ: Give a direct proof of theorem 16.1.10 without using theorem . 16.1.9.

16.1.12: Theorem: The product of any non-empty class of connected spaces isconnected.

Proof: Let $\{X_i\}$ be a non-empty class connected spaces. Let X = $\prod_i X_i$ be the product pace of the topological spaces $\{X_i\}$ space that X is connected. it is enough to prove that any Continuous function from X into the discrete two point space {0,1} is not onto.
Let f : X→ {0,1} be a continuous function.
Part A: We first prove that if two elements of X differ in atmost one componentthen they have the same image under the mapping *f*,
Let a = $\{a_i\}$ and x = $\{x_i\} \in$ X and let it be an index such that x_i = ai for $i \neq i_1$
Define $f_{i_1} : x_i \to X$ by $f_{i_1}(t) = \{y_i\}$ where $Y_{i_1} = t$ and $y_i = a_i$ for $i \neq f_{i_1}$ is continuous mapping from $X_{i_1} \to X$.
Now $fof_{i_1} : X_{i_1} \to \{0.1\}$ is continuous. Since X_{i_1} is connected fof_{i_1} is a constant
Now $fof_{i_1}\{a_{i_1}\}$ = f(a) *and* f(x) = $fof_{i_1}\{x_{i_1}\}$. fof_{i_1} is a constant imply that
$fof_{i_1}\{a_{i_1}\} = fof_{i_1}\{x_{i_1}\}$. Thus f(x) = f(a).

B. let a = $\{a_i)$ be a fixed element of X. Now we prove that if x∈X and X differsfrom a in atmost n components then f(x) = f(a).

If n =1 then the result is true by part A. Suppose that the result is true for n = k. let x∈X such that x differs from a in almost k+1 components, say $i_1, i_2, \ldots\ldots i_k, i_{k+1}$.
Define y =$\{y_i\}$∈X by $y_{i_f} = x_{i_f}$ for f = 1,2,….,k, and $y_i = a_i$ for $i \neq i_1, i_2, \ldots\ldots, i_k$.
Then x and y differ in at most in their $i_{(k+1)}$ th component.
∴By Pan A we have f (x) = f(y). Also y and a differ at most in their $i_1, i_2, \ldots\ldots i_k$. components.
By induction hypothesis we have f (y) = f (a).
∴f (x) = f (a). Hence the result is true for all n.

C. Fix some a∈X.
Let A={x∈X / x differs from a in atmost a finite number of components}.

Then it can be shown that A is a dense subset of X. Also by pan B. f is aconstant on A. Since {0,1} is a T_I - space, we get that f is a constant mapping on X. Hence f is not onto. Thus there is no continuous mapping of X onto the discrete two point space {0,1}. Hence X is connected.

16.1.13: SAQ: Let X be a topological space and let Y be a T_I – space. Letf : X→ Y be a continuous map such that f is a constant on a dense subset A of X. Prove that f is constant on X.

16.1.14: Theorem: A subspace of the real line R is connected if. and only if it isan interval. In particular R is connected.

Proof. Let X be a subspace of R. Suppose X is connected. To show X is an interval
1) Suppose that X is not an interval. Then there exist real numbers r, s, t ∋
r < s < t, r, t ∈X and s∉X. The sets A = X∩(– ∞ , s) and B = X∩(s, + ∞) are non–empty disjoint open sets in X such that X = A∪B
Hence X is not connected .
2) Assume that X is not connected. Let X = A∪B be a disconnection of X. Then A and B are

non-empty and disjoint closed, as well as open, subsets of X. We can choose $x \in A$ and $z \in B$ such that $x \neq z$. We may assume that $x < z$. Now $x,z \in X$. $[x.z] \cap A$ is bounded above by z. Hence $y=\sup([x.z] \cap A)$ exists in R. It is clear that $x \leq y \leq z$, Since X is an interval. $x,z \in X$. we have $y \in X$. Since A is also closed in X, the definition of Y shows that $y \in A$.

$\therefore$ $y < z$. Also if $\in > 0$ then $y < y + \in < z$ implies $y + \in \in B$. Since B is closed in X we get $y \in B$. Thus $y \in A \cap B$, which is a contradiction since A and B are disjoint.

Hence X is connected

The proof is complete from (1) and (2).

16.1.15: Theorem: The range of a continuous real valued function on a connected space is an interval.

Proof. Let $f : X \to R$ be a continuous real valued function. Let $Z = f(X)$.

By theorem 16.1.10, Z is connected. By theorem 16.1.14, we get that Z is an interval. Theorem16.1.15may also be stated as follows: Let f be a real valued continuous mapping on a connected space X. Let $x,y \in X$. Let c be a real number $\ni f(x) \leq c \leq f(z)$. Then $\exists z \in X \ni f(z)=c$. Thus theorem 16.1.15 is also called "Intermediate value theorem".

16.1.16. Theorem : The spaces R^n and C^n are connected.

Proof : We know that R. being an interval, is connected with the usual topology. We also know that R^n as a topological space can be regarded as the product of n copies of the connected space R. Hence by theorem 16.1.12.

We get that R^n is connected. We show that C^n and R^{2n} are homeomorphic as topological spaces.

Let $z =(z_1 ,z_2 ,...,z_n) \in C^n$, Let $z_k = a_k +ib_k$ for $k = 1,2,\ldots,n$.

Define $f:C^n \to R^{2n}$ by $f(z) = (a_1.b_1,a_2.b_2\ldots\ldots a_n.b_n)$

Clearly f is one-one and onto and $|f(z)| = |z|$.

Thus f is an isometry of C^n onto R^{2n} and hence f is a homeomorphism. Since R^{2n} is connected we have C^n is connected.

16.2: SHORT ANSWER QUESTIONS :

16.2.1: Prove that $X = A \cup B$ is a disconnection of a topological space X iff A and B are non-empty disjoint closed sets.

16.2.2: Show that a topological space X is connected if, for and only if, every non-empty proper subset of X has non-empty boundary.

16.2.3: Show that a topological space X is connected if, and only if, for every two points in X there is some connected subspace of X which contains both.

16.2.4: Prove that a subspace of a topological space X is disconnected iff it can be represented as the union of two non-empty sets each of which is disjoint from the closure in X of the other.

16.2.5: Show that the graph of a continuous real function defined on an interval is a connected subspace of the Euclidean plane.

16.2.6: If X is a countable. connected topological space. show that constant functions are the only real valued continuous functions on X.

(Hint. Use Theorem 15.1.15 and the fact that every interval with more than one point in R is uncountable).

16.2.7: Determine whether the following are connected subspaccs of R^2

i. $\{(x, y) \in R^2 / x \neq 0\}$
ii. $\{(x, y) \in R^2 / x^2 + y^2 = 1\}$
iii. $\{(x. \mathrm{Sin}\,(1/x) / 0 \neq x \in R)$
iv. $\{(x,y) \in R^2 / x \neq y\}$

16.2.8: For any completely regular space X. prove that X is connected iff the Stone- Cech compactification $\beta(X)$ of X is connected.

16.2.9: 1f T_1 and T_2 are topologies on X such $T_1 \subseteq T_2$ and (X_1, T_1) is connectedprove that (X_2, T_2) is also connected.

16.2.10: Prove that a topological space X is connected iff every continuousfunction from X into the discrete two point space {0, l} is constant

16.3 SUMMARY:

We learrnt that the connected spaces and connected topological space. We have proved that the range of a continuous real valued function on a connected space is an interval.

16.4 TECHNICAL TERMS:

1. **Topological space;** A set with a topology.
2. **Topology;** A collection of open sets.
3. **Open set;** A set that contains all its limit points.
4. **Connected space;** A space that cannot be written as the union of two disjoint non – empty open sets.
5. **Discrete topology;** A topology where every set is open.

16.5 ANSWER TO SELF ASSESSMENT QUESTIONS

16.1.5. SAQ. T= $\{\phi, X\}$. Thus X is the only non-empty open set and hence X can not be represented as the union of two non-empty disjoint open sets.

Hence (X,T) is a connected space.

16.1.6:SAQ: Let $a \in X$. Then $B = X \backslash \{a\}$ is non-empty. Since T is the discrete topology.Every subset of X is open in (X,T). Thus $X = \{a\} \cup B$ is a disconnection of X.

16.1.11:SAQ: Let f: $X \to Y$ be a continuous function and suppose X is connected.
Let Z=f (X).
Let $Z = A \cup B$ be a disconnection of Z.
Then $\exists$ open sets G and H in Y such that $A = Z \cap G$ and $B = Z \cap H$.
Let $G_1 = f^{-1}(H)$ and $H_1 = f^{-1}(H)$. Then G_1 and H_1 are open in X.
$A \neq \phi \Rightarrow \exists\, x \in X \ni f(x) \in A = Z \cap G$
$\Rightarrow f(x) \in G \Rightarrow X \in f^{-1}(G) = G_1$

similarly $\exists\, y \in X \ni y \in f^{-1}(H) = H_1$. Thus $G_1 \neq \phi$ and H_1

$$t \in G_1 \cap H_1 \Rightarrow f(t) \in G \text{ and } f(t) \in H$$
$$\Rightarrow f(t) \in Z \cap G = A \text{ and } f(t) \in Z \cap H = B$$
$$\Rightarrow f(t) \in A \cap B \text{ which is a contraditction.}$$

$\therefore G_1 \cap H_1 = \phi$

$$t \in X \Rightarrow f(t) \in Z$$
$$\Rightarrow f(t) \in A \text{ or } f(t) \in B$$
$$\Rightarrow t \in f^{-1}(G) \text{ or } t \in f^{-1}(H)$$
$$\Rightarrow t \in G_1 \text{ or } t \in H_1$$
$$\Rightarrow t \in G_1 \cup H_1$$

Thus X = $G_1 \cup H_1$ is a disconnection of X, which contradicts the hypothesis that X is connected. Hence Z is connected.

16.1.13: SAQ : Let f(x) = $\forall x \in A$, since Y is a T_I– space and $a \in Y$, {a} is closedin Y.

Since f is continuous $f^{-1}(\{a\})$ is closed in X. We have

$$A \subseteq f^{-1}(\{a\}) \Rightarrow \bar{A} \subseteq \overline{f^{-1}(\{a\})} = f^{-1}(\{a\})$$

$$\Rightarrow X = \bar{A} \subseteq f^{-1}(\{a\})$$

$$\Rightarrow f(X) = \{a\}$$

Thus f is a constant map on X.

16.6 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

Dr. M. Gnaneswara Reddy

LESSON – 17
THE COMPONENTS OF A SPACE

OBJECTIVES:

The objectives of this lesson are to.

- To understand the concepts of components of a topological space.
- To understand the concepts of a connected topological space.
- To understand the concepts of a connected subspace of a topological space.

STRUCTURE:

17.0: INTRODUCTION:

In this lesson, we will prove that a topological space X can always be de-composed into a disjoint union of maximal connected subspaces of X, which wecall the components of X.

17.1: CONECTED SPACE:

17.1.1: Definition : Let (X. T) be a topological space. A connected subspace ofX is said to be a component of X if A is not properly contained in any other connected subspace of X. That is, a sub space A of X is a component if it is connected. and $A \subseteq B, B$ connected implies A = B.

17.1.2: If X is a connected space. then X is the only component of X.

17.1.3: In a discrete topological space X. any set with more than one element isdisconnected. Hence singleton sets are the only components of X.

17.1.4: Example: In the space Q of rational numbers with usual topology, any subspace A with more than one element is not connected; for if r, s∈A and r < s we can find an irrational t ∋ r < t < s and $A = [A \cap (-\infty, t)] \cup [A \cap (t + \infty)]$ is a disconnection of A. Thus singleton sets are the only components of Q. But the usual topology in Q. is not discrete.
We prove the following two theorems before we attempt to decompose a space into its components.

17.1.5: Theorem: Let X be a topological space, let $\{C_i\}$ be a non-empty class of connected subspaces of X such that $\cap C_i$ is non-empty. Then the subspace$C = \cup C_i$ is connected.

Proof: Suppose $C \subseteq A \cup B$ here A and B are open sets in X such That is $A_1 = C \cap A$ and $B_1 = C \cap B$ are disjoint. For each i, the connected set $C_i \subseteq C$ and hence $C_i \subseteq A \cup B$. $(C_i \cap A) \cap (C_i \cap B) \subseteq A_1 \cap B_1 = \phi$. Since C_i is connected by Lemma 16.1.4. either $C_i \cap A = \phi$ or

$C_i \cap B = \phi$. Thus $C_i \subseteq A$ or $C_i \subseteq B$. Since $\cap C_i \neq \phi$ we have either all the $C_i = \phi$ are contained in A or all the C_i are contained in B. Thus $C = \cup C_i \subseteq A$ or $C = \cup C_i \subseteq B$ Hence $C \subseteq A$ or $C \subseteq B$. Thus $C \cap B = \phi$ *or* $C \cap A = \phi$ .Hence C is connected.

17.1.6 : Theorem : Let A be a connected subspace of a topological space X. Let B be a subspace of X such that $A \subseteq B \subseteq \bar{A}$. Then B is connected in particular is connected.

Proof : Assume that B is disconnected. Then $\exists$ open sets G and H of X such that
$B \subseteq G \cup H, G_1 = B \cap G \neq \phi, H_1 = B \cap H \neq \phi$ and $G_i \cap H_i = \phi$
Since $A \subseteq B \subseteq G \cup H$ and A is connected, either $A \subseteq H$ and $A \cap G = \phi$ or $H_1 = B \cap H \neq \phi$ .Suppose $A \cap G = \phi$. Then $\bar{A} \cap G = \phi$. If $A \cap H = \phi$ then $\bar{A} \cap H = \phi$. Since $B \subseteq \bar{A}$ we get that $B \cap G = \phi$ or $B \cap H = \phi$ which is a contradiction. Thus B is connected.
$\therefore A \subseteq \bar{A} \subseteq \bar{A}$, by above $\bar{A}$ is connected.

17.1.7: Theorem : Let X be a topological space. Then we have the following

(i) Each point of X is contained in exactly one component of X.

(ii) Each connected subspace of X is contained in a component of X.

(iii) Each component of X is closed in X

(iv) A connected subspace of X which is both open and closed is a component of X.

Proof: (i) Let $x \in X$. Let A= { $C \subseteq X$/ $x \in C$ and C is connected subspace of X}
Then A= ϕ since $\{x\} \in A$ and $x \in \cap_{C \in A} C$
By theorem 17.1.5 $C_x = \cap_{C \in A} C$ is connected.
If D is any connected subspace of $X \ni C_x \subseteq D$ then $x \in D$. So, D is in the class A. Hence D$\subseteq C_x$
$\therefore D = C_x$
Thus C_x is a component of X. If E is any component of $X \ni x \in E$ then $E \subseteq C_x$
Since E is a component and C_x is connected we have $C_x = E$.

(i) Let C be a connected subspace of X. If $x \in C$ then $C \subseteq C_x$

(ii) Let C be a component. Since C is connected, by theorem 17.1.6. $\bar{C}$ is connected $C \subseteq \bar{C}$ and C is a component $\Rightarrow C = \bar{C} \Rightarrow C$ is closed.

(iii) Let C be a connected subspace which is both open and closed in X. By (ii)$\exists$ a component E $\ni C \subseteq E$. Then C is open and closed in E also. Since E is connected we have C= ϕ or C = E. Since C is a subspace, we have C $\neq$Q, Hence C = E is a component.

17.1.8: SAQ: Prove that a topological space X is connected if, and only if, X hasno non empty proper subset which is both open and closed.

17.1.9: SAQ: If the product $\prod X_i$ is connected prove that each X_i is connected

17.10: SAQ: Is a component of X Open in X?

17.1.11: SAQ. Prove that the components of a space form u partition of X. If thereare only a finite number of components of a space X. prove that each componentis open.

17.2: MODEL EXAMINATION QUESTIONS:

1) Define a connected space and prove that a topological space X is connected iff there is no continuous function from X onto the discrete two point space (0,1).
2) Prove that the product of any non-empty class of connected spaces is connected.
3) Describe connected subsets of the real line **R**.
4) Prove that the continuous image of a connected space is also a connectedspace, Prove that $\mathbf{R}^n$ and C^n are connected.
5) Define a component of a topological space. What are the components of **Z**. the set of all integers. as a subspace of the real line **R** with the usual topology?
6) If $A \subseteq B \subseteq \overline{A}$ for subs paces A and B of a topological space X and A is connected, show that B is also connected.

Prove that the components of a topological space X are closed subset of X. Can we prove that components of X are also open subsets of X ? Justify your answer.

17.3: EXCERCISES :

1) Let $\{C_i\}$ be a non-empty class of connected subspace of a topological space X such that $C_i \cap C_j \neq \varphi$ for all r and J. Prove that $\cup C_i$ is also connected. (Hint: proof of theorem 17.1.5)
2) Let A_1. A_2 ..... A_n ... be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1} \neq \varphi$ for n = 1, 2,...... prove that $\cup_{n=1} A_n$ is connected. (Hint let $B_n = \cup_I^n A_k$. Then $\cup_I^\infty A_n = \cup_I^\infty B_n$. Use induction to prove B_n is connected and use theorem 17.1.5 with the class $\{B_n\}$).
3) Use theorem 17.1.5 to prove that X x Y is connected if X and Y are connected.
4) Prove that an open subspace of the complex plane is connected if, and onlyif. any two points in it can be joined by a polygonal line.

17.4 SUMMARY:

We learrnt that the connected space, connected subspace and topological space. We have proved that Let X be a topological space, let $\{C_i\}$ be a non-empty class of connected subspaces of X such that $\cap C_i$ is non-empty. Then the subspace$C = UC_i$ is connected.

17.5 TECHNICAL TERMS:

1. **Connected space;** A topological space that cannot be written as the union of two disjoint non-empty open sets.
2. **Connected subspace;** A subspace of a topological space that is connected.
3. **Topological space;** A set equipped with a topology, which is a collection of open sets.
4. **Component;** A maximal connected subset of a topological space.

17.6 ANSWER TO SELF ASSESSMENT QUESTIONS:

17.1.8: SAQ: Suppose X is connected. If A is a non-empty proper subset of X which is both open and closed then $X = A \cup (X \setminus A)$ would form a disconnection of X. If $X = A \cup B$ is a disconnection of X then A (and also B) is a non-emptyproper subset of X which is both open and closed in X.

17.1.9: SAQ: Hint. Use theorem 16.1.10 with the projection mapping $P_i: X \rightarrow Xi$.

17.1.10: SAQ: See example 12.34. Singleton sets are not open in Q. since if [r]$\subseteq$ Q is open in Q then { r } $\supseteq$ Q$\cap$ (a. b) for some open internal (a, b) in **R**. ButQ$\cap$ (a.b) has infinitely many points.

17.1.11: SAQ: Let X be a topological space. Each $x \in X$ belongs to a unique component C_x

Then $= \bigcup_{x \in X} C_x$. If $C_x \cap C_y \neq \varphi$ then C= $C_x \cup C_y$ is connected $C_x \subseteq C$ and $C_y \subseteq C$ simply C = $C_x = C_y$. Thus thecomponents of X from a partition of X.

Let C_1 , C_2 ....C_n , be the only distinct components of X. The $X = \bigcup_{i=1}^{n} C_i$ and each Cx being a component. is closed.

For each i, $D_i = \bigcup_{\substack{j=1 \\ j \neq 1}}^{n} C_j$ is closed and hence $C_i = X - D_i$ is open.

17.7 SUGGESTED READINGS:

1. Introduction to Topology and Modern Analysis by G.F. Simmons, McGraw-Hill Book Company, New York International student edition.

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